

Gauge Fluxes in F-theory and Type IIB Orientifolds

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ABSTRACT:

We provide a detailed correspondence between G_4 gauge fluxes in F-theory compactifications with $SU(n)$ and $SU(n) \times U(1)$ gauge symmetry and their Type IIB orientifold limit. Based on the resolution of the relevant F-theory Tate models we classify the factorisable G_4 -fluxes and match them with the set of universal D5-tadpole free $U(1)$ -fluxes in Type IIB. Where available, the global version of the universal spectral cover flux corresponds to Type IIB gauge flux associated with a massive diagonal $U(1)$. In $U(1)$ -restricted Tate models extra massless abelian fluxes exist which are associated with specific linear combinations of Type IIB fluxes. Key to a quantitative match between F-theory and Type IIB is a proper treatment of the conifold singularity encountered in the Sen limit of generic F-theory models. We also collect evidence that the Type IIB orientifold limit of the considered Tate models involves a non-trivial B -field and shed further light on the brane recombination process relating generic and $U(1)$ -restricted Tate models.

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1 Introduction

Our understanding of four-dimensional F-theory [1–3] vacua as compactifications on singular elliptic fibrations supplemented by consistent G_4 -fluxes has improved considerably over the past few years. Apart from representing interesting examples of non-perturbative string dynamics in their own right, F-theory compactifications on singular Calabi-Yau four-folds have attracted a lot of recent attention in the context of 4-dimensional GUT model building [4–8]. The singularities of the internal elliptic Calabi-Yau four-fold Y_4 are in 1-1 correspondence with the gauge groups along 7-branes [9], the matter fields at the intersection of 7-branes [10] and their Yukawa interactions [4, 5, 11], as reviewed e.g. in [12, 13]. Of particular importance for a well-defined dimensional reduction is the resolution of these singularities in terms of a smooth Calabi-Yau four-fold \hat{Y}_4 . Inspired by $SU(5)$ GUT model building, fully-fledged Calabi-Yau four-folds and their resolutions have been constructed via toric methods in [14, 15] and [16–18] (see [19] for the construction of a base space of such fibrations). Subsequently, a more detailed analysis of higher codimension singularities in $SU(5)$ models has been provided in the resolution schemes of [20–23].¹

1.1 The Quest for Gauge Fluxes

In this article we focus on a better understanding of the flux sector of F-theory compactifications. Gauge fluxes are key to model building because they are responsible, among other things, for a chiral matter spectrum. Various types of chirality inducing gauge fluxes have been constructed recently in terms of G_4 -fluxes defined on a resolved four-fold \hat{Y}_4 [21–23, 30]. While this marks an important step forward in F-theory model building, a number of open questions concerning the flux sector remain. This has to do with the fact that G_4 -fluxes are given by harmonic four-forms in $H^{2,2}(\hat{Y}_4)$, while the gauge flux along a 7-brane W is thought of as a two-form $F \in H^{1,1}(W)$. In fact, most of the information on the gauge flux F is encoded only very implicitly in G_4 , and the precise relation between fluxes in F-theory and Type IIB is not obvious. Typical open questions concern the localization of fluxes on individual branes, the appearance of D-terms or the role of the D5-brane tadpole [31]. Our aim is to systematize the construction of gauge fluxes in $SU(n)$ and $SU(n) \times U(1)$ F-theory models and to gain a better intuition for these fluxes by relating them to gauge fluxes in the Type IIB limit. Along the way we will also make contact with the semi-local description of gauge fluxes via spectral covers [4, 8, 14, 32–35]. Somewhat surprisingly, we will identify the universal spectral cover fluxes with specific fluxes in Type IIB.

Before summarizing our main findings in section 1.2, we now set out to provide some more background on the construction of fluxes in Type IIB, via spectral covers and as G_4 -fluxes.

The probably most intuitive and familiar way to think about gauge theories on 7-branes is in terms of a Type IIB orientifold on a Calabi-Yau X_3 with stacks of 7-branes along holomorphic divisors. Chirality inducing gauge fluxes arise as the $U(1)$ -fluxes associated with the diagonal $U(1)$ of the $U(N)$ gauge groups realised on the 7-branes at generic

¹Recent progress concerning six-dimensional F-theory vacua has been achieved e.g. in [24–29].

position. To the extent that each such brane stack comes with its own $U(1)$ -gauge potential and thus its own set of fluxes, gauge fluxes are localised on the branes. A more mathematical way to phrase this is that they are simply the first Chern class of the $U(1)$ -bundles over the various 7-branes. The flux degrees of freedom assemble, at first sight, into a vector $F \in H^{1,1}(D_A)$ on each brane D_A . Being related to a $U(1)$ -symmetry, each such flux induces a field dependent Fayet-Iliopoulos term (in the usual abuse of nomenclature) entering the D-term condition for the respective 7-brane. While chiral indices at the brane intersection depend only on the flux difference between the intersecting branes, topological invariants such as the 3-brane tadpole are sensitive to the local flux data along each individual 7-brane.

If we uplift a Type IIB model to F-theory we can also consider the semi-local neighbourhood of one of the non-abelian brane stacks, thereby making contact with the spectral cover approach to F-theory model building. A generic, say, $SU(5)$ model is described semi-locally by a Higgs bundle over the GUT brane S , whose structure group is the commutant $SU(5)_\perp$ within an underlying E_8 gauge group, $E_8 \rightarrow SU(5) \times SU(5)_\perp$ [4, 8, 32, 35]. Naively the associated fluxes seem of a completely different nature than the ones in Type IIB models: First, the role of the complementary $SU(5)_\perp$ is *sans pareil* in Type IIB. Also the number of flux degrees of freedom do not seem to match: The spectral cover fluxes are described in terms of a two-form in $\eta \in H^2(S)$, but this two-form is completely fixed once the geometry is specified. The degrees of freedom of these fluxes are merely one overall discrete parameter to be chosen in such a way that the fluxes are well-quantised. To confuse us even more, since there is no extra $U(1)$ not contained within $SU(5)$, the fluxes do not induce any D-term, again in contrast to Type IIB expectations. Does this mean that such spectral cover fluxes are truly non-perturbative and do not exist in Type IIB at all? For non-generic, so-called split spectral covers, another set of fluxes arises [14, 32–35]. These do induce a D-term and are thus closer to the IIB picture, but it is not clear at all where they are localized - if that notion is appropriate in the first place.

In global F-theory compactifications gauge fluxes are encoded, via duality with M-theory, by G_4 -fluxes.² As anticipated above, these are specified by certain elements of $H^{2,2}(\hat{Y}_4)$ subject to a number of constraints which will be reviewed at the beginning of section 3.1. In this picture the localization of gauge fluxes familiar from Type IIB models has become completely obscure. A notable exception are fluxes associated with the Cartan $U(1)$ s of a non-abelian gauge group G along a divisor \mathcal{W} on the base: The corresponding four-forms are of the type $F \wedge w_i$, where the two-forms $w_i, i = 1, \dots, \text{rk}(G)$ are associated with the \mathbb{P}^1 s in the fiber over \mathcal{W} needed to resolve the singularity (more precisely, they are Poincaré dual to the divisor obtained by fibering these over \mathcal{W}), and $F \in H^{1,1}(B)$. This agrees with Type IIB intuition if we identify $F|_{\mathcal{W}}$ with the flux along the 7-brane. A recent discussion in particular of such Cartan fluxes has been given in [39]. For non-Cartan fluxes, however, which are the fluxes that we are interested in here, the geometric picture sketched above does not apply, and an identification of G_4 with a flux F along some 7-brane is difficult.

²In addition, G_4 -fluxes describe the analogue of bulk fluxes in Type IIB, which are relevant for moduli stabilization. See e.g. [36–38] for an incomplete list of recent investigations of various aspects of such fluxes and their superpotentials in F-theory.

1.2 Summary of Results

To match G_4 gauge fluxes with the analogous objects in Type IIB orientifolds and in the language of spectral covers, we will work in a specific type of F-theory models. Concretely, we construct the Tate model and its resolution corresponding to an $SU(n)$ gauge group along a divisor W as well as the $U(1)$ -restricted Tate model [40] leading to gauge group $SU(n) \times U(1)$. Our first task in section 2.1 is to compute the resolution divisors, their detailed intersection structure and the topological invariants of the resolution space, thereby generalising our previous analysis in [22], which was valid for $n = 5$. This serves two purposes: First, it provides the necessary topological data to quantitatively compare the F-theory geometry to its Type IIB weak coupling limit; second, the construction of G_4 -fluxes hinges upon control over the four-forms of the resolved space \hat{Y}_4 . We then analyze the Type IIB limit of this class of F-theory models in section 2.2. The pure $SU(n)$ models correspond to a Type IIB orientifold with one $U(n)$ brane stack and its image as well as one invariant 7-brane of Whitney-brane type [41]. In the $SU(n) \times U(1)$ model, the latter splits into a brane/image-brane pair. When considering the Sen limit, an interesting complication arises that had first been observed in [32] for $SU(5)$ models and in fact holds more generally: The Type IIB three-fold X_3 associated with a generic F-theory model with $SU(n)$ gauge group exhibits a conifold singularity. For $SU(5)$ models this singularity is related to the existence of the E_6 -point at which the **10 10 5** Yukawa couplings are realised. In the presence of such a conifold singularity, F-theory and Type IIB are not smoothly connected. Our strategy is to impose certain constraints on the topology of the models such that this singularity is absent. Indeed for such models typical topological data such as the Euler characteristic on the one hand and the curvature induced 3-brane charge on the other match, as is verified in section 2.3.

Having established a solid geometric foundation we can analyze the F-theory G_4 gauge fluxes in our class of models. On a Calabi-Yau four-fold \hat{Y}_4 there are two types of harmonic $(2,2)$ -forms: four-forms which factorise into the wedge product of two-forms and those which do not. The subspace of $H^{2,2}(\hat{Y}_4)$ spanned by linear combinations of factorisable four-forms is called primary vertical subspace $H_{\text{vert.}}^{2,2}(\hat{Y}_4)$ [42, 43]. In section 3.1, we classify the primary vertical G_4 -fluxes on the resolution four-folds associated with our $SU(n)[\times U(1)]$ models. For generic $SU(n)$ models, i.e. for models with no extra $U(1)$ -factor, no such fluxes exist if $n < 5$ apart from the $SU(n)$ Cartan fluxes. By contrast, for $SU(n) \times U(1)$ models one can always construct gauge flux associated with the extra $U(1)$ as in [22, 23, 30, 40]. These are related to a class of non-factorisable fluxes in generic $SU(n)$ -models via brane recombination. Beginning with $n = 5$, as another class of fluxes we recover the so-called universal fluxes first observed in [21], which in fact correspond to universal spectral cover fluxes under heterotic duality. We work out the detailed topological signatures such as induced chiralities and 3-brane charges in section 3.2 and comment on the quantisation condition for the fluxes. For definiteness we focus on restricted $SU(5) \times U(1)$ and generic $SU(5)$ models. In particular, we find agreement with the (split) spectral cover approach in section 3.3.

In section 4 we compare the G_4 -fluxes to gauge fluxes in Type IIB. Finding match

between the two pictures relies crucially on the D5-tadpole cancellation condition. In F-theory, this condition is already built in, while in Type IIB models it must be imposed by hand and significantly reduces the number of consistent fluxes. For the two types of brane setups corresponding to the $SU(n)$ and the $SU(n) \times U(1)$ models, we identify, in section 4.1, a generating set for all universally present gauge fluxes satisfying the D5-tadpole cancellation condition and find only two such inequivalent types of fluxes. We further analyze these fluxes in section 4.2 by specifying to $(S)U(5) \times U(1)$. One of the two sets of fluxes corresponds to the gauge flux associated with the linear combination of $U(1)$ s that remains massless with respect to the geometric Stückelberg mechanism. All its topological characteristics such as D3-charge, chiral spectrum and the D-term match with the $U(1)$ -flux in $SU(n) \times U(1)$ F-theory models. In particular this explains the apparent “delocalisation” of this type of G_4 flux because the corresponding Type IIB fluxes are a linear combination of fluxes on both brane stacks. In addition, we identify a special gauge flux associated with the geometrically massive diagonal $U(1) \subset U(n)$. For $n = 5$ this flux matches exactly the universal spectral cover flux, which is rather surprising given the very different origin of this flux in the two pictures. This means that the fluxes used in heterotic spectral cover models and the diagonal Type II fluxes are really the same objects. We stress, though, that our conclusions hold in this direct form for the special type of models with no conifold singularity and thus no E_6 -point. This E_6 -point would violate the perturbative Type IIB selection rules associated with the diagonal $U(1)$. The identification of the diagonal $U(1)$ -flux with the spectral cover flux is also surprising in view of the analysis of [31], according to which the diagonal $U(1)$ is decoupled from the low-energy spectrum and described by certain non-harmonic two-forms. In section 4.3 we put our findings in perspective with the analysis of [31]. In section 4.2 we also compare the flux quantization condition in Type IIB and F-theory models. We provide evidence that if the GUT divisor is non-spin, the Type IIB orientifold must involve a non-trivial B-field. This is consistent with the results of [44] obtained for Weierstrass models with up to $SU(2)/Sp(1)$ gauge group.

Finally, in section 5 we shed some more light on the brane recombination process that relates the restricted $SU(n) \times U(1)$ and the generic $SU(n)$ Tate models. Gauge fluxes can obstruct this recombination if the spectrum of recombination modes is purely chiral. We analyze necessary conditions for recombination to be possible in the presence of fluxes and match these with restrictions on the gauge flux on the recombined side as found previously in [30].

Some open questions are summarized in section 6. Most of the technicalities that occurred in the course of our analysis have been relegated to the appendices.

2 F-theory Four-folds Versus Type IIB Brane Configurations

In this section we construct the class of F-theory four-folds and their Type IIB limits which serves as our laboratory to compare the respective gauge fluxes. Starting with the singular F-theory fibrations corresponding to $SU(n)$ - or $SU(n) \times U(1)$ gauge theories, we calculate characteristic topological invariants that allow us to quantitatively match these geometries

with their Type IIB orientifold limit. We then derive the analogous brane configurations in Type IIB theory by taking the Sen limit and find agreement between the geometric D3-tadpole contributions in both pictures. As we will see this requires a proper treatment of certain conifold singularities encountered in the Sen limit.

2.1 The Geometry of F-theory $SU(n)(\times U(1))$ Models

The geometry of a four-dimensional F-theory compactification is given by an elliptically fibered four-fold Y_4 . We define the four-fold as a divisor in an ambient five-fold X_5 by describing it via a Weierstrass model in Tate form

$$P_T = \{y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6\}. \quad (2.1)$$

The coordinates (x, y, z) are homogenous coordinates of the fibre ambient space $\mathbb{P}_{2,3,1}$ fibered over a three-dimensional base B . The Tate polynomial coefficients $a_i(u_i)$ depend on local coordinates on B such as to form sections of \bar{K}^i , where \bar{K} denotes the anti-canonical bundles of B .³

As is well-known the 7-brane locus in this set-up is given by the vanishing of the discriminant of the Weierstrass model. In models with an $SU(n)$ -singularity in the fiber over a base divisor $\mathcal{W} : w = 0$ the discriminant factorises as

$$\Delta = w^n \Delta'. \quad (2.2)$$

This is achieved by restricting the sections a_i in (2.1) in a manner determined by application of the Tate algorithm [9]. The vanishing orders of a_i along $w = 0$ are collected in eq. A.2 in the appendix. In such models the fibre over $w = 0$ degenerates to an \tilde{A}_{n-1} -singularity. In absence of further gauge groups the fibre above generic points on Δ' acquires merely an I_1 -singularity. The singularity type of the fibre enhances over the intersection curves of the divisor $w = 0$ and the I_1 -locus, which is also where massless matter charged under $SU(n)$ is located.

In the sequel we assume that the divisor $\mathcal{W} : w = 0$ itself is smooth and connected. In particular this excludes the possibility that the discriminant locus self-intersects in a curve contained in \mathcal{W} . Such self-intersections host extra matter - here in the symmetric representation of $SU(n)$ - and have been analyzed recently in [45].

It may be phenomenologically preferable to have additional $U(1)$ -symmetries in the F-theory model, e.g. to engineer specific selection rules in $SU(5)$ based GUT models. A certain class of models with such extra abelian gauge symmetries is provided by fibrations with extra sections. As an example of such geometries we consider here, in addition to the above $SU(n)$ model, the $U(1)$ -restricted Tate model, obtained by the additional requirement that the Tate polynomial coefficient a_6 vanish everywhere on the base [40]. This introduces an additional $SU(2)$ -singularity along a self-intersection curve of the divisor Δ' .

³Cohomology classes on the base B of the F-theory four-fold will be denoted by caligraphic letters. For simplicity we use the same symbols for the base classes and their pull-back to the four-fold as well as for classes in cohomology and their Poincaré dual homology classes.

Since the enhanced singularity of the $SU(n)$ or $SU(n) \times U(1)$ model renders the entire four-fold Y_4 singular (rather than just the fibre), special care must be taken in calculating topological invariants on it. It is easiest to resolve the singularities first and then perform our calculations on the resulting smooth resolution manifold \hat{Y}_4 .⁴ The resolution can be done by pasting in (possibly weighted) \mathbb{P}^2 s into the ambient space, which reduce to \mathbb{P}^1 s on the Calabi-Yau four-fold located exactly at the fibre's singular points. In particular, to resolve the $SU(n)$ singularity one introduces a set of so-called exceptional divisors e_i along with a scaling relation for each e_i ,

$$(x, y, e_0, e_i) \sim (\lambda^{v_3} x, \lambda^{v_4} y, \lambda e_0, \lambda^{-1} e_i), \quad (2.3)$$

where $e_0 = 0$ denotes the proper transform of $w = 0$. For $SU(n)$ -singularities one can show that the powers v_3, v_4 are intrinsically related to the vanishing orders of the Tate polynomial coefficients a_3, a_4 . We leave the details to appendix A.1. The intersection structure of the various E_i (the divisor classes of the divisors $\{e_i = 0\}$ on \hat{Y}_4) is directly related to the $SU(n)$ -Cartan matrix via

$$\int_{\hat{Y}_4} E_i E_j \mathcal{B}_a \mathcal{B}_b = C_{ij} \int_{\mathcal{W}} \mathcal{B}_a \mathcal{B}_b, \quad (2.4)$$

where \mathcal{B}_k are base divisor classes.

In the $SU(n) \times U(1)$ model, one must resolve in addition the mentioned curve of $SU(2)$ -singularity along the base curve

$$C_{34} = \{a_{3,v_3} = 0\} \cap \{a_{4,v_4} = 0\}. \quad (2.5)$$

To resolve this via blow-up, a \mathbb{P}^1 is pasted in along the submanifold $\{x = 0\} \cap \{y = 0\}$ in the ambient space, $(x, y) \rightarrow (\tilde{x}s, \tilde{y}s)$. Reducing to the four-fold, only those \mathbb{P}^1 s which are fibred over C_{34} remain. The blow-up introduces another divisor $S : s = 0$, which is actually a section, together with a new scaling relation $(x, y, s) \sim (\lambda x, \lambda y, \lambda^{-1} s)$.

We should stress an important point: The above construction of a resolved four-fold \hat{Y}_4 works irrespective of the details of the base space B - provided B has enough sections such as to form a Tate model in the first place *without creating any singularities apart from the ones accounted for in the Tate model itself*. This is a condition that must be checked in concrete examples. The advantage of this method is that it allows us to reduce all expressions in our analysis to general quantities defined directly on the base space B . This is key for a comparison with Type IIB orientifolds on the double-cover of B via the Sen limit. Alternatively, one may directly construct the singular four-fold Y_4 fibered over a concrete base B and resolve it via toric methods, as in [14–17, 23]. While this is particularly powerful for explicit model building, base independent computations are less immediate to perform.

In the remainder of this section we sketch the logic behind the computation of the intersection forms on \hat{Y}_4 and of topological quantities such as $c_2(\hat{Y}_4)$ and $\chi(\hat{Y}_4)$. All details are provided in appendix A.

⁴Alternatively one can apply the technology of singular cohomology and continue to work on Y_4 .

Since the blow-up divisors as well as the fibre \mathbb{P}_{231} form a toric subspace of the ambient manifold, it is possible to use toric methods to deduce certain relations concerning the intersection structure of these divisors. In particular, one can derive a base-independent subset of the generator set of non-intersecting brane configurations of the ambient space and, in combination with the proper transform of the Tate polynomial, compute a subset of the Stanley-Reisner ideal of the Calabi-Yau four-fold. The Stanley-Reisner ideal encodes the sets of non-intersecting divisors. For example, in all elliptic fibrations of the form described above, xyz is always an element of the Stanley-Reisner ideal, indicating that the three divisors $\{x = 0\}$, $\{y = 0\}$, $\{z = 0\}$ do not intersect in the ambient space. Put differently, there is no patch for this manifold on which all three of these variables are allowed to vanish.

From the various elements of the Stanley-Reisner ideal, it is in turn possible to express double intersections of two exceptional divisors as linear combinations of double intersections involving base divisors. In particular, the double intersections of two $SU(n)$ resolution divisors are expressible as

$$E_i E_j = C_{ij} (Z + \bar{K}) \mathcal{W} + w_m E_m \mathcal{W} + k_m E_m \bar{K} + b E_2 E_4, \quad (2.6)$$

where $b = 0$ for $SU(n)$ models with $n < 5$. The details of the derivation, and in particular the coefficients w_m and k_m are provided in appendix A.2. In all cases the coefficients of the first term are the $SU(n)$ Cartan matrix entries.

The relations just described reduce the number of independent products of two-forms. This property will be useful later on, when we consider potential flux configurations for F-theory models. They are also useful in order to express the second Chern class of the Calabi-Yau four-fold, which enters the flux quantisation condition, as well as to express the fourth Chern class, which is needed to evaluate the Euler characteristic. The latter two points are what we focus on here.

From the scaling relations described in (2.3) and the general adjunction formula, one can deduce the relationship between the Chern class of the resolution manifold and the Chern class of the original manifold. We here outline the generic $SU(n)$ model. More details as well as a derivation of the analogous quantities in the $SU(n) \times U(1)$ model can be found in appendix A.3.

The scaling relation (2.3), introduced by each blow-up, implies that in each case the class of the divisor $\{x = 0\}$ changes according to $X \rightarrow \pi^* X - v_3^i E_i$, where π^* denotes the pullback of the original class to the resolution manifold. Similarly one can read off the changes for the divisors $\{y = 0\}$, $\{e_0 = 0\}$. All other divisors remain in the pullbacks of the original classes, while the class of the vanishing locus of the Tate divisor changes by $-(v_3^i + v_4^i) E_i$. The sum of the changes is zero, which, along with the adjunction formula, implies that the first Chern class of the resolution manifold will be the same as that of the original manifold. This demonstrates that the resolution manifold is indeed Calabi-Yau, if we start with a Calabi-Yau manifold. These properties then also imply that the change in the second Chern class can be expressed in terms of the above changes. We leave the explicit expressions for the $SU(n)$ -cases with $n \leq 5$ to appendix A.3.

From the expressions for the change of the second Chern class one can in turn derive the change in the Euler characteristic of the four-fold. Since the arithmetic genus $\chi_0 = \frac{1}{720} \int_{Y_4} (c_4 - 3(c_2)^2) = 2$ is the same for all Calabi-Yau four-folds (see e.g. [44], where this was exploited in a similar context), the change in c_4 is related to the change in c_2 by

$$\Delta c_4 = 3 \left(2 c_2^{ns} \Delta c_2 + (\Delta c_2)^2 \right). \quad (2.7)$$

Here we define

$$\Delta c := c \left(Y_4^{\text{resolved}} \right) - c \left(Y_4^{\text{non-singular}} \right) \quad (2.8)$$

and c_2^{ns} is the pullback of the second Chern class of the original, non-singular four-fold to the resolution manifold. In fact, this is given by $c_2^{ns} = 12(Z + \bar{K})^2 + [c_2(B_3)] + \bar{K}^2$. Noting that, in the $SU(n)$ -cases, the change in the second Chern class is perpendicular to Z as well as to the intersection of two base divisors, it is clear that the first term in the above equation vanishes. Then in each of the $SU(n)$ -cases, the change in the Euler characteristic reduces to an integral over the GUT-brane. The precise formulae are again collected in appendix A.3. Here we note that we reproduce the spectral cover formulae of [14] for $SU(n)$, $n = 2, 3, 5$, and provide a similar formula for $SU(4)$, which is not derivable via spectral covers. The $SU(5)$ -case has already been computed in [21]. The total Euler characteristic of the resolution manifold is then simply the sum of $\Delta\chi$ and the well-known Euler characteristic of the non-singular, elliptically fibred Calabi-Yau four-fold

$$\chi_{ns} = 360 \bar{K}^3 + 12 \bar{K} [c_2(B)]. \quad (2.9)$$

The analogous expressions for restricted $SU(n) \times U(1)$ models are derived by the same logic and are listed in appendix A.3, see in particular table 12 for the change in $c_2(\hat{Y}_4)$ in that case. Note furthermore that the Euler characteristic drops by an additional

$$\Delta_{su_n}^{su_n \times u_1} \chi(Y_4) = 3 \chi(C_{34}) \quad (2.10)$$

as a consequence of the $U(1)$ -restriction.

2.2 Sen Limit and Conifold Points

The connection between the F-theory $SU(n)$ -models of the previous sections and their perturbative formulation as Type IIB orientifolds is made via the well-known Sen limit. As we will discuss, in order to carry out this limit without encountering singularities we must impose certain restrictions on the models under consideration. These will turn out crucial also for a correct identification of the F-theory and Type IIB gauge fluxes.

Generalities of the Sen limit

Recall from [46] that for an elliptic fibration in Weierstrass form

$$y^2 = x^3 + f x z^4 + g z^6 \quad (2.11)$$

with f and g sections of \bar{K}^4 and \bar{K}^6 , respectively, the orientifold limit corresponds to letting $\epsilon \rightarrow 0$ in the parametrisation

$$f = -3h^2 + \epsilon \eta, \quad g = -2h^3 + \epsilon h \eta - \frac{\epsilon^2}{12} \chi. \quad (2.12)$$

In this limit the discriminant locus $\Delta = 4f^3 + 27g^2$ factorises as

$$\Delta = -9\epsilon^2 h^2 (\eta^2 - h\chi) + \mathcal{O}(\epsilon^3). \quad (2.13)$$

To apply this to a Weierstrass model in Tate form as in (2.1) we note that the sections f and g are related to the Tate polynomial a_i via

$$f = -\frac{1}{48}(b_2^2 - 24b_4), \quad g = -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6) \quad (2.14)$$

with

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6. \quad (2.15)$$

To perform the Sen limit one therefore identifies [32]

$$b_2 = -12h, \quad b_4 = 2\epsilon\eta, \quad b_6 = -\frac{\epsilon^2}{4}\chi, \quad (2.16)$$

corresponding to a rescaling

$$a_3 \rightarrow \epsilon a_3, \quad a_4 \rightarrow \epsilon a_4, \quad a_6 \rightarrow \epsilon^2 a_6. \quad (2.17)$$

The discriminant takes the form

$$\Delta = -\epsilon^2 b_2^2 b_8 + \mathcal{O}(\epsilon^3), \quad b_8 = \frac{1}{4}(b_2b_6 - b_4^2). \quad (2.18)$$

The Calabi-Yau three-fold X_3 on which the Type IIB orientifold is defined takes the form of the double cover of the F-theory base manifold B . It is given by the hypersurface equation

$$X_3 : \xi^2 = b_2, \quad (2.19)$$

where $b_2(u_i)$ depends on the local coordinates u_i of B . The orientifold involution acts as

$$\sigma : \xi \longrightarrow -\xi. \quad (2.20)$$

The O7-plane therefore corresponds to the fix point locus $b_2 = 0$, while the vanishing locus of b_8 represents the 7-brane configuration.

The Calabi-Yau three-fold is related to the F-theory base B through a two-to-one map

$$\pi : X_3 \rightarrow B, \quad (2.21)$$

see also Figure 1. This induces a natural action on (co)homology. In particular, to each element $\omega_B \in H^2(B)$ we can assign a two-form $\omega_{X_3} \in H^2(X_3)$ by pullback,

$$\omega_{X_3} = \pi^*(\omega_B). \quad (2.22)$$

Finding the exact relation between the forms on X_3 and B is part of the task of constructing an F-theory uplift of a given Type IIB model. There are two types of behaviour for a divisor class on X_3 under the orientifold involution σ : It can be mapped to itself or it can

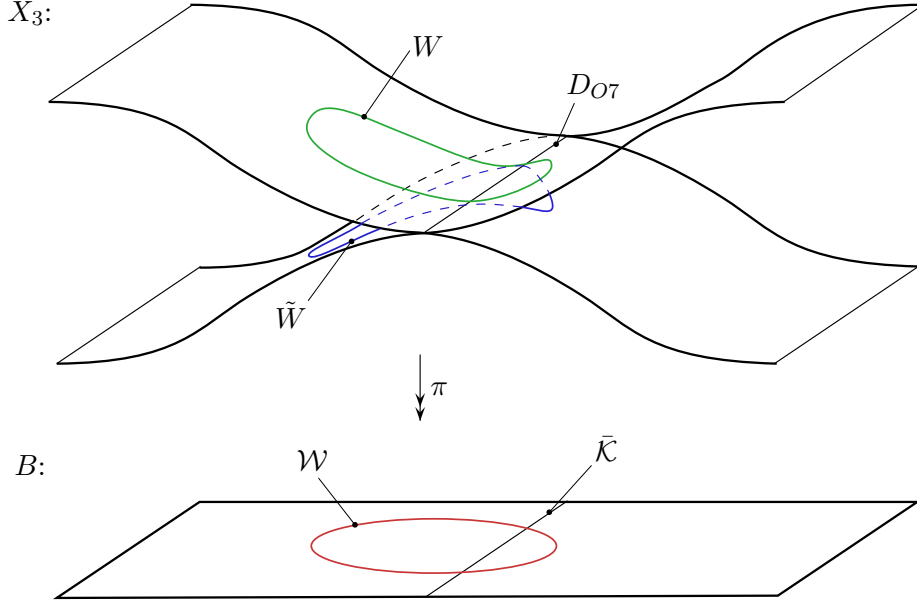


Figure 1. Schematic drawing of the Calabi-Yau X_3 and its orientifold projection π to B . The Calabi-Yau is embedded as a hypersurface in the $\mathcal{O}(\bar{\mathcal{K}})$ -bundle over B .

be exchanged with another class. The F-theory uplift of such exchange involutions has previously been considered in [47–49]. If the orientifold involution σ on X_3 exchanges two classes D and \tilde{D} it is useful to introduce the combinations

$$D_{\pm} = D \pm \tilde{D}. \quad (2.23)$$

The class D_+ survives the projection from X_3 to B . The preimage of D_+ under π^* in $H^2(B)$ will then be denoted by \mathcal{D} , such that

$$\pi^*(\mathcal{D}) = D_+. \quad (2.24)$$

In addition, one must determine the precise relation between invariant classes on X_3 and their analogue on B . For example the Sen limit implies that

$$\pi^*(\bar{\mathcal{K}}) = [D_{O7}], \quad (2.25)$$

where D_{O7} is the class of the orientifold plane. This allows one to deduce further relations between invariant classes on X_3 and their descendants on B . The intersection numbers of elements in $H^2(B)$ and their pullbacks on X_3 are related as

$$\int_{X_3} \pi^*(\mathcal{D}_a) \wedge \pi^*(\mathcal{D}_b) \wedge \pi^*(\mathcal{D}_c) = \int_{\pi(X_3)} \mathcal{D}_a \wedge \mathcal{D}_b \wedge \mathcal{D}_c = 2 \int_B \mathcal{D}_a \wedge \mathcal{D}_b \wedge \mathcal{D}_c, \quad (2.26)$$

where the factor of 2 arises because ‘ $\pi(X_3) = 2B$ ’. Note that this is consistent with the explicit uplift models considered in [49] and the general analysis in [31].

Sen limit for $SU(n)$ models and conifold points

We are now ready to apply this reasoning to the $SU(n)$ Tate model of the previous section by specifying the vanishing behavior $a_i = a_{i,v_i} w^{v_i}$ encoded in eq. (A.2) in the appendix. On general grounds, in Type IIB orientifolds an $(S)U(n)$ gauge symmetry is realized by a brane/image-brane pair exchanged by the orientifold involution. Therefore the 7-brane stack on the divisor $w = 0$ on the F-theory base B must correspond to a stack/image-stack pair on X_3 . Similarly, for the restricted Tate model with $a_{6,n} = 0$ one expects another brane/image-brane pair corresponding to the extra $U(1)$ appearing in the theory.

To see how this comes about we first observe that one can re-express b_8 as follows, where we distinguish whether n is even or odd,

$$b_8 = \begin{cases} w^n \left[a_{6,n} b_2 - \left(a_{4,k} + \frac{a_1 + \xi}{2} a_{3,k} \right) \left(a_{4,k} + \frac{a_1 - \xi}{2} a_{3,k} \right) \right], & n = 2k, \\ w^n \left[a_{6,n} b_2 - w \left(a_{4,k+1} + \frac{a_1 + \xi}{2w} a_{3,k} \right) \left(a_{4,k+1} + \frac{a_1 - \xi}{2w} a_{3,k} \right) \right], & n = 2k + 1. \end{cases} \quad (2.27)$$

Note that for $n = 2k + 1$ the second term in square brackets is no longer symmetric if one pulls the w -factor into one of the two round brackets. On the other hand, as pointed out, one would expect that in the restricted case ($a_{6,n} = 0$) b_8 factors into w^n times two factors which are exchanged under the involution so as to represent a brane/image-brane pair. To remedy this apparent problem, we note that the divisor $w = 0$ does indeed split into two on X_3 : From the Calabi-Yau equation

$$P_{X_3} = (\xi - a_1)(\xi + a_1) - a_{2,1}w = 0 \quad (2.28)$$

it follows that $P_{X_3}|_{w=0}$ factorises. Let us thus define

$$\omega = \{w = 0\} \cap \{\xi - a_1 = 0\}, \quad (2.29)$$

$$\tilde{\omega} = \{w = 0\} \cap \{\xi + a_1 = 0\}. \quad (2.30)$$

These two divisors are exchanged under the involution $\xi \longleftrightarrow -\xi$. Then ω and $\tilde{\omega}$ define the brane/image-brane pair as expected to account for $SU(n)$ gauge symmetry. Note that, while in the ambient space they both lie in the same divisor class, on the Calabi-Yau three-fold X_3 they generically lie in different classes.

Since, on the three-fold, the section w factors into two components, it is possible to write the second term in square brackets of the odd- n -case of eq. (2.27) as

$$\left(a_{4,k+1}\omega + \frac{a_1 + \xi}{2\tilde{\omega}} a_{3,k} \right) \left(a_{4,k+1}\tilde{\omega} + \frac{a_1 - \xi}{2\omega} a_{3,k} \right). \quad (2.31)$$

Now the two terms are exchanged under the involution, leading to a brane/image-brane pair in the $U(1)$ -restricted case. Note that, similarly to the GUT brane/image-brane pair, the brane and its image do not necessarily lie in the same class.

One might object that ω and $\tilde{\omega}$ appear in the denominator of the above expressions, and that these are therefore not everywhere well-defined. However, from the definitions of ω , $\tilde{\omega}$ and the Calabi-Yau equation one can see that the expression $\frac{\xi - a_1}{\omega}$ corresponds to the algebraic cycle given by the intersection of $\{\xi - a_1\}$ and $\{a_{2,1}\}$ in the ambient space,

and similarly $\frac{\xi+a_1}{\tilde{\omega}}$ corresponds to $\{\xi + a_1\} \cap \{a_{2,1}\}$. Hence, we obtain two well-defined expressions for the additional branes in the case of a splitting.

We now come to an important subtlety. For general base manifolds B the Calabi-Yau equation (2.28) exhibits a conifold singularity at [32]

$$a_1 = a_{2,1} = w = 0. \quad (2.32)$$

In the presence of such a conifold point, it is not possible to smoothly interpolate between the F-theory picture and the Type IIB orientifold regime. In particular, it is not clear that topological invariants should agree on both sides. Therefore, we only consider the Sen limit in cases where this intersection is not realised on the base. Note that for $n = 5$ this conifold point is precisely the point of E_6 singularity enhancement [32] at which the top Yukawa couplings **10 10 5** of an $SU(5)$ model are located, which are perturbatively absent in Type IIB models. We thus demand that we do not have an “ E_6 ”-point (or its generalizations in general $SU(n)$ models) on the F-theory side. This enforces the special relation

$$2 \int_B \bar{\kappa}^2 \mathcal{W} = \int_B \bar{\kappa} \mathcal{W}^2. \quad (2.33)$$

The corresponding relation on the IIB side is that

$$\int_{X_3} D_{O7} (2 D_{O7} - W_+) W_+ = \int_{X_3} D_{O7} (2 D_{O7} - W - \tilde{W})(W + \tilde{W}) = 0. \quad (2.34)$$

In fact, from equation (2.28) we observe that the loci $\{\xi = 0\} \cap \{\omega = 0\}$ and $\{\tilde{\omega} = 0\} \cap \{\omega = 0\}$ are identical. Written in terms of classes, this becomes

$$D_{O7} W = W \tilde{W} = D_{O7} \tilde{W} \quad \Rightarrow \quad W_+^2 - W_-^2 = 2 D_{O7} W_+. \quad (2.35)$$

Note that (2.35) implies (2.34). Therefore, F-theory models which arise as the uplift of smooth Type IIB orientifolds do automatically not admit the above type of conifold points, as was observed in the examples of [48, 49]. We will see in the next section that the conditions (2.35) and (2.33) are crucial for a quantitative match between the topological invariants of the F-theory and the Type IIB models.

Let us summarize the brane set-ups as the Type IIB limits of the $SU(n)$ - and $SU(n) \times U(1)$ - F-theory models: The brane contents corresponding to the $SU(n)$ F-theory models, with generic non-vanishing sections $a_{6,n}$, is that of a $U(n)$ brane/image-brane stack along W and \tilde{W} together with a single connected 7-brane along the divisor in square-brackets in (2.31). This latter divisor is singular and of so-called Whintey type. Whitney branes of this form have been analyzed in detail in [41] and further in [30, 44]. In appendix B.2 we provide a generalisation of the description of [30, 44], which is in the context of models with extra non-abelian gauge groups up to $Sp(1)/SU(2)$, to general $SU(n)$ models. We will find some novelties which are important for a quantitative match with F-theory. Since the Whitney brane carries trivial gauge group, the gauge group in Type IIB is $SU(n) \times U(1)_a$, where

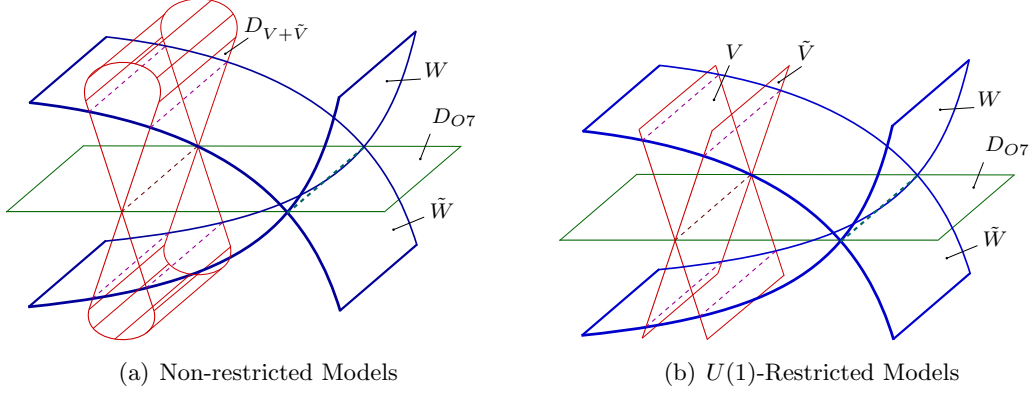


Figure 2. Brane configurations for (a) $SU(n)$ - and (b) $SU(n) \times U(1)$ - Type IIB set-ups. Note that V and \tilde{V} lie in the same class in the latter case if n is even.

the latter denotes the diagonal abelian factor in $U(n)$. This $U(1)_a$ is massive by means of the Stückelberg mechanism since the stack and image-stack are in different homology classes on X_3 . The models are therefore precisely of the type considered recently in [31].

On the other hand, for the $U(1)$ -restricted Tate models, with $a_6 \equiv 0$, the extra brane in square-brackets in (2.31) splits into a brane-image brane pair. If $n = 2k$, both are in the same homology class, whereas for $n = 2k + 1$ they are not. The special situation of $SU(2)$ has already been analyzed in [30]. The gauge group in the first case is $SU(n) \times U(1)_a \times U(1)_b$, where both $U(1)$ -factors are individually massive, but a massless linear combination of them exists. For n odd, the second $U(1)_b$ -factor is by itself massless.

A schematic drawing of the various brane set-ups is given in figure 2. In both cases, the GUT stack is situated on the divisor $\{\omega = 0\}$ and its image on $\{\tilde{\omega} = 0\}$. The remaining brane of the non-restricted case lies in the class $D_{V+\tilde{V}} = 8D_{O7} - n(W + \tilde{W})$ as is required to cancel the $D7$ -charge induced by the O-plane. In the restricted case, the remaining brane splits into a brane/image-brane pair with associated divisor classes V and \tilde{V} . For n even, these two classes are the same, $V = \tilde{V} = \frac{1}{2}D_{V+\tilde{V}}$, however the divisors themselves do not necessarily coincide. On the other hand, in the odd case V and its image are different classes: $V = 4D_{O7} - kW - (k+1)\tilde{W}$ where k is defined via $n = 2k + 1$.

2.3 Topological Invariants of Type IIB Brane Configurations

As a first and important check of the Sen limit of our F-theory $SU(n)$ model we compute the induced D3-brane charges of the Type IIB 7-branes and orientifold plane and compare them with the Euler characteristic of the resolved F-theory four-fold \hat{Y}_4 .

Recall that the D3-brane tadpole cancellation condition for a Type IIB orientifold takes the form

$$Q_{\text{gauge, IIB}} + N_{D3} = \frac{1}{4}N_{O3} + Q_{D3}^g, \quad (2.36)$$

where $Q_{\text{gauge, IIB}}$ represents the gauge flux induced 3-brane charge, N_{D3} and N_{O3} counts the number of D3-branes and O3-planes and Q_{D3}^g is the curvature induced 3-brane charge from the O7-plane and the D7-branes. In this section we focus on the latter. The contribution of

a brane/image-brane pair to Q_{D3}^g depends on the Euler characteristic $\chi(D_i)$ of the smooth divisor D_i ,

$$Q_i^g = \frac{n_i}{48} \left(\chi(D_i) + \chi(\tilde{D}_i) \right), \quad \chi(D) = \int_{X_3} \left(D^3 + c_2 D \right). \quad (2.37)$$

For an invariant brane along a divisor of Whitney type, on the other hand, the singular geometry must be resolved and the 3-brane charge is proportional to the Euler characteristic of the resolved space [41],

$$Q_i^g = \frac{1}{48} \chi_0(D_i) = \frac{1}{48} \int_{X_3} \left(D_i^3 + c_2 D_i - 3 D_i D_{O7} (D_i - D_{O7}) \right). \quad (2.38)$$

Note that the expressions (2.37) and (2.38) are often referred to as the "downstairs" picture because we compute all quantities for each brane and its image (or, in the case of the Whitney brane, for the entire invariant brane on X_3) and divide by 2.

The geometric tadpole contribution for non-restricted $SU(n)$ -models is then given by

$$24 Q_{D3}^g = \frac{n}{2} \left(\chi(W) + \chi(\tilde{W}) \right) + \frac{1}{2} \chi_0(8D_{O7} - n W_+) + 2 \chi(D_{O7}). \quad (2.39)$$

The first term is due to the $SU(n)$ stack and its image, the second due to the Whitney brane and the last term represents the contribution from the O7-plane. This is readily evaluated and to be compared with $\chi(\hat{Y}_4)$ of the resolved F-theory four-fold as given by the sum of (2.9) and the results collected in table 13. Note that to obtain match we crucially need the relations (2.34) and (2.35). With the help of the identification (2.25) and (2.24) as well as the factor $\frac{1}{2}$ in (2.26) we find agreement for the D3-tadpole for $SU(n)$, $n = 3, 4, 5$.

In a similar manner one finds perfect match between the Euler characteristic of the $U(1)$ -restricted $SU(n)$ Tate models and their Sen limit. More details are provided in appendix A.4.

$SU(2)$ versus $Sp(1)$

As an aside we note that the above expressions do not quite match in the $SU(2)$ -case; they would if $W_+ (W_+^2 - W_-^2)$ were zero in this case. To understand the mismatch, we calculate the D3-charge of a brane set-up which corresponds to an $Sp(1)$ -gauge theory. This consists of a stack of two branes in some invariant divisor class W and a Whitney brane in the class $8D_{O7} - 2W$. The corresponding tadpole formula,

$$24 Q_{D3}^g = \frac{n}{2} \chi(W) + \frac{1}{2} \chi_0(8D_{O7} - n W) + 2 \chi(D_{O7}), \quad (2.40)$$

(with $n = 2$) leads to an expression which matches the F-theory D3-tadpole upon the usual identifications.

This suggests an identification of the $SU(2)/Sp(1)$ F-theory model with the $Sp(1)$ - rather than the $SU(2)$ -brane set-up in Type IIB. This roots in the resolution algorithm as presented in section 2.1. Everything worked out there relies on divisor classes and is therefore 'blind' to cases where the hypersurface is restricted to a special form. However,

this happens for $SU(2)$ where a_2 is restricted to $a_{2,1}w$ although a general a_2 would be allowed by the multi-degree of P_T .

For Calabi-Yau manifolds embedded as complete intersections in a toric variety we can make this more explicit. There the patch structure of the ambient space can be encoded in terms of a (N-lattice) polytope and its triangulation. Each lattice point inside this polytope corresponds to a toric divisor of the ambient space. From this one can construct a dual (M-lattice) polytope encoding the monomials of the hypersurface. Restricting the Tate polynomial coefficients, $a_i = a_{i,k}w^k$, translates into removing points from this dual polytope. The convex hull of the remaining set of points then forms a smaller dual polytope. If this is reflexive, the dual serves as an ambient space in which the singularities of the hypersurface are resolved. By comparing this to the original manifold, one can read off the additional divisors of the ambient space.

Now, moving from an $Sp(1)$ - to an $SU(2)$ -singularity corresponds to restricting the second Tate polynomial coefficient to $a_2 = a_{2,1}w$. Since both singularities are of rank one, one does not expect an additional divisor from this restriction. Furthermore, since the restrictions necessary to induce an $Sp(1)$ -singularity are also required to induce an $SU(2)$ -singularity, the $SU(2)$ -dual polytope has to be contained within the $Sp(1)$ -dual polytope. In other words, those points of the $Sp(1)$ -dual polytope which are removed in moving to an $SU(2)$ -singularity are either interior points with no effect on the actual ambient space or they are such that the resulting $SU(2)$ -dual polytope is not reflexive any more. In both cases the deformation cannot be resolved by the canonical procedure. We thus conclude that the above derivation for the D3-tadpole contribution of the F-theory model does actually correspond to an $Sp(1)$ -model rather than an $SU(2)$ -model.

3 Gauge Fluxes in F-theory $SU(n)$ and $SU(n) \times U(1)$ Models

In this section we classify the set of factorisable gauge fluxes in F-theory models with $SU(n)$ and $SU(n) \times U(1)$ gauge symmetry. Specialising for concreteness to $n = 5$ we provide a number of characteristic topological invariants which will allow us to compare these fluxes both to their spectral cover description and, in section 4, to make contact with fluxes in the Type IIB limit. For a discussion of G_4 -gauge fluxes in the recent literature on globally defined four-folds see [21–23, 30].

3.1 ‘Vertical’ G_4 -Fluxes in $SU(n)(\times U(1))$ F-theory Models

Gauge flux in F-theory is described in terms of the flux associated with the field strength $G_4 = dC_3$ of the dual M-theory dimensionally reduced on \hat{Y}_4 . A first constraint on G_4 is that it must be vertical, i.e. orthogonal both to the section $Z : z = 0$ of \hat{Y}_4 and to the pullback of any two base divisors \mathcal{B}_a ,

$$\int_{\hat{Y}_4} G_4 \wedge Z \wedge \mathcal{B}_a = 0, \quad \int_{\hat{Y}_4} G_4 \wedge \mathcal{B}_a \wedge \mathcal{B}_b = 0. \quad (3.1)$$

Some further consistency conditions must be met. First, according to [50] G_4 must be (half-)integer quantised

$$G_4 + \frac{1}{2}c_2(\hat{Y}_4) \in H^4(\hat{Y}_4, \mathbb{Z}). \quad (3.2)$$

The F-term supersymmetry condition requires that in addition $G_4 \in H^{2,2}(\hat{Y}_4)$. In the presence of abelian gauge symmetries extra D-term conditions arise, as will be discussed further below.

As anticipated in the introduction, a Calabi-Yau four-fold admits two different types of such elements of $H^{2,2}(\hat{Y}_4)$. Forms that can be expressed as the sum of the wedge product of two elements of $H^{1,1}(\hat{Y}_4)$ are denoted as elements of the primary vertical subspace $H_{\text{vert}}^{2,2}(\hat{Y}_4)$ [42, 43]. Such fluxes automatically satisfy the F-term supersymmetry condition for all choices of complex structure moduli and therefore exist everywhere in complex moduli space. Typically, the vast majority of four-forms cannot be written as the sum of such factorisable four-forms. In this section we wish to classify the available gauge fluxes in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$ for the $SU(n)$ and $SU(n) \times U(1)$ resolution four-forms. In a first step this requires finding a basis of factorisable four-forms satisfying (3.1). In a second step we discuss which of these leave the non-abelian part of the gauge group invariant so that they can be switched on in phenomenological models e.g. of $SU(5)$ GUT type to induce a chiral spectrum without affecting the gauge symmetry.

$SU(n)$ -Models

An obvious set of fluxes in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$ that satisfy (3.1) is given by the Cartan fluxes, i.e. four-forms of the type $[E_i] \wedge [\mathcal{F}_i]$ for some base divisor classes $\mathcal{F}_i \in H^{1,1}(B)$. Another possibility are combinations of $[E_i] \wedge [E_j]$ -terms which are orthogonal to the intersection of two base divisors. Orthogonality to $Z\mathcal{B}_a$ follows immediately from the Stanley-Reisner ideal elements ze_i , derived in appendix A.2. On the other hand, the intersection of $[E_i] \wedge [E_j]$ with two base divisors always involves the Cartan matrix entries C_{ij} , cf. (2.6). It is thus easy to see that potential flux candidates are given by linear combinations of $[E_i] \wedge [E_j]$ whose corresponding Cartan-matrix entries add up to zero. As an example consider $SU(5)$. In view of the specific form of $SU(n)$ -Cartan matrix,

$$C_{ij} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}, \quad (3.3)$$

viable combinations include for instance E_2E_4 or $E_2E_3 - E_3E_4$, while for example E_2E_3 is not a viable flux candidate.⁵

It is simple enough to derive a basis of linearly independent viable combinations, and we list one such basis for $SU(n), n \leq 5$ in table 15 in the appendix. From the derivations

⁵If the divisor \mathcal{W} has vanishing intersection with every other divisor class, more vertical fluxes exist. Clearly this situation is not of interest to applications and will thus be discarded.

in subsection 2.1, in particular from (2.6), one notes that the forms listed therein can be re-expressed entirely in terms of Cartan fluxes $E_i \bar{\mathcal{K}}$, $E_i \mathcal{W}$ and – in the case of $SU(5)$ – in terms of $E_2 E_4$. In particular, none of the combinations depends on $(Z + \bar{\mathcal{K}})\mathcal{W}$, because the coefficients of this term in (2.6) are just the Cartan matrix elements, and we have chosen combinations whose Cartan matrix elements add up to zero. Since there is no linear combination of Cartan fluxes that is orthogonal to all possible Cartan fluxes, linear combinations of $E_i E_j$ -terms do not add new elements to the space of potential fluxes for $SU(n)$ with $n < 5$. This, in turn, implies that there are no factorisable gauge fluxes leaving the $SU(n)$ gauge symmetry intact for these $SU(n)$, $n = 2, 3, 4$.

For the $SU(5)$ -case, on the other hand, there appears one additional element in (2.6) and we had previously chosen to use $E_2 E_4$. With the help of the intersection numbers

$$\int_{\hat{Y}_4} E_2 E_4 E_i \mathcal{B}_a = \int_{\mathcal{W}} (1, -1, 1, -1)_i \bar{\mathcal{K}} \mathcal{B}_a, \quad \int_{\hat{Y}_4} E_2 E_4 E_2 E_4 = \int_{\mathcal{W}} -\bar{\mathcal{K}} \mathcal{W} \quad (3.4)$$

we can then use this element in combination with Cartan fluxes to arrive at a combination of four-forms which is orthogonal to all Cartan fluxes,

$$G_4^\lambda = \lambda \left(E_2 E_4 + \frac{1}{5} (2, -1, 1, -2)_i E_i \bar{\mathcal{K}} \right). \quad (3.5)$$

This flux has been noted previously in [21], albeit with a different derivation. Our overall normalisation has been chosen for later convenience. We use the λ -label in anticipation of the fact that this flux is identified with a so-called universal spectral cover flux, whose parameter space simply consists of one scaling parameter. The discrete parameter λ will later on be constrained such that the flux quantisation condition is satisfied.

$SU(n) \times U(1)$ -Models

For Weierstrass models with the additional restriction $a_{6,n} = 0$, the resolution of the induced singularity along C_{34} results in an additional divisor class S . Its intersection properties were analyzed in [22] and include two particularly useful properties,

$$\int_{\hat{Y}_4} (S - Z - \bar{\mathcal{K}}) Z \mathcal{B}_a \mathcal{B}_b = 0, \quad \int_{\hat{Y}_4} (S - Z - \bar{\mathcal{K}}) \mathcal{B}_a \mathcal{B}_b \mathcal{B}_c = 0. \quad (3.6)$$

Thus, it is clear that expressions of the type $[S - Z - \bar{\mathcal{K}}] \wedge [\mathcal{F}]$ (where $[\mathcal{F}]$ is a two-form of the base) form a set of additional vertical flux candidates. In order to find additional fluxes that preserve the gauge group, this can be combined with the usual Cartan fluxes to construct linear combinations which are orthogonal to all Cartan fluxes. For the various gauge groups and corresponding resolution manifolds considered here, we list the results in table 16 in the appendix.

For future reference it is useful to define w_X to be precisely this linear combination in each case:

$$w_X = (S - Z - \bar{\mathcal{K}}) + t^i E_i \quad (3.7)$$

with t^i given in table 16. Again, the rationale behind our overall normalisation, which differs from the one we used in [22] by a factor of $\frac{1}{n}$, will become clear later. We note that

the condition for this to be orthogonal to all Cartan fluxes translates into

$$\delta_{jN} + t^i C_{ij} = 0 \quad (3.8)$$

where N is the index of the exceptional divisor which intersects the additional divisor $\{s = 0\}$, see also the definition below table 5 in the appendix. This property is useful in the analysis of the D3-tadpole contribution, chirality, or D-term induced by this type of flux.

3.2 Specialisation to G_4 Fluxes in $SU(5)(\times U(1))$ Models

For concreteness we restrict ourselves in the following to the analysis of $SU(5) \times U(1)$ -models and, towards the end of this section, to generic $SU(5)$ models.

As established above, a basis for factorisable gauge fluxes compatible with the $SU(5) \times U(1)$ gauge symmetry is given by the two fluxes $G_4^X(\mathcal{F}) = -w_X \wedge \mathcal{F}$ and G_4^λ ,

$$G_4^X(\mathcal{F}) = -(S - Z - \bar{\mathcal{K}} + \tfrac{1}{5}(2, 4, 6, 3)_i E_i) \wedge \mathcal{F}, \quad (3.9)$$

$$G_4^\lambda = \lambda (E_2 \wedge E_4 + \tfrac{1}{5}(2, -1, 1, -2)_i E_i \wedge \bar{\mathcal{K}}). \quad (3.10)$$

Note again that fluxes of the type G_4^λ do not exist for $SU(n)$, $n = 2, 3, 4$. The flux $G_4^X(\mathcal{F})$, on the other hand, is associated with the extra abelian gauge factor, which we call $U(1)_X$ in the sequel. To better understand the nature of these fluxes and to compare them to gauge fluxes in Type IIB orientifolds we now compute the induced chiralities, D3-tadpole contribution and D-terms of these fluxes.

Chiralities in $SU(5) \times U(1)$ models

The chiral spectrum in models with G_4^X has already been computed in [22, 23]. The matter spectrum of the $U(1)$ -restricted $SU(5)$ GUT model consists of chiral multiplets in representations $\mathbf{10}_1$, $\mathbf{5}_3$, $\mathbf{5}_{-2}$, $\mathbf{1}_{-5}$, where the subscripts denote the $U(1)_X$ charges. These states in representation R_q arise from membranes wrapping certain combinations of resolution \mathbb{P}^1 s in the fibre over the matter curves C_{R_q} in the base. To each component of the representation one can assign a matter surface $S_{R_q}^k$ by fibreing the corresponding combination of \mathbb{P}^1 s over C_{R_q} . See also [20, 21] for a description of these matter surfaces for non-restricted $SU(5)$ models and [23] for a toric approach.

The chirality induced by G_4^X is then computed by integrating the flux over the corresponding matter surfaces [4]. The result is identical for each component of the representation. For G_4^X this can be shown to yield [22, 23]

$$\chi(R_q) = \int_{S_{R_q}^k} G_4^X(\mathcal{F}) = \frac{q}{5} \int_{C_{R_q}} \mathcal{F}. \quad (3.11)$$

In order to compare these expressions to the Type IIB-picture later on, we also list the classes of divisors whose intersection defines the various matter curves inside the base:

$$\begin{aligned} C_{\mathbf{5}_3} &= \mathcal{W}(3\bar{\mathcal{K}} - 2\mathcal{W}), & C_{\mathbf{10}_1} &= \mathcal{W}\bar{\mathcal{K}}, \\ C_{\mathbf{5}_{-2}} &= \mathcal{W}(5\bar{\mathcal{K}} - 3\mathcal{W}), & C_{\mathbf{1}_{-5}} &= (4\bar{\mathcal{K}} - 3\mathcal{W})(3\bar{\mathcal{K}} - 2\mathcal{W}). \end{aligned} \quad (3.12)$$

By the same strategy the chiral index induced by the universal flux G_4^λ can be computed. This requires the intersection numbers derived in the previous sections. We collect the chiralities of both types of fluxes in table 1.

State	Chirality under G_4^λ	Chirality under $G_4^X(\mathcal{F})$
10₁	$\frac{1}{5}\lambda \int_{C_{10_1}} -6\bar{\mathcal{K}} + 5\mathcal{W}$	$\frac{1}{5} \int_{C_{10_1}} \mathcal{F}$
5₃	$\frac{1}{5}\lambda \int_{C_{5_3}} 2\bar{\mathcal{K}}$	$\frac{3}{5} \int_{C_{5_3}} \mathcal{F}$
5₋₂	$-\frac{1}{5}\lambda \int_{\mathcal{W}} \mathcal{W}\bar{\mathcal{K}}$	$-\frac{2}{5} \int_{C_{5_{-2}}} \mathcal{F}$
1₋₅	0	$-\int_{C_{1_{-5}}} \mathcal{F}$

Table 1. Chiral index for massless matter. In the case of G_4^λ the chirality for **5₋₂** was derived via anomaly cancellation.

D3-tadpole in $SU(5) \times U(1)$ models

The D3-tadpole cancellation condition in F-theory takes the well-known form

$$Q_{\text{gauge,F}} + N_{D3} = \frac{\chi(\hat{Y}_4)}{24}, \quad Q_{\text{gauge,F}} = \frac{1}{2} \int_{\hat{Y}_4} G_4 \wedge G_4. \quad (3.13)$$

The flux contribution $Q_{\text{gauge,F}}$ is easily evaluated with the intersection numbers at hand,

$$Q_{\text{gauge,F}}(G_4^X) = - \int_B \left[(\bar{\mathcal{K}} - \frac{3}{5}\mathcal{W}) \mathcal{F}^2 - \frac{1}{5}\lambda \bar{\mathcal{K}} \mathcal{W} \mathcal{F} + \frac{1}{2}\lambda^2 \bar{\mathcal{K}} \mathcal{W} (\frac{6}{5}\bar{\mathcal{K}} - \mathcal{W}) \right]. \quad (3.14)$$

The D-Term in $SU(5) \times U(1)$ models

Since the $U(1)$ -restricted Tate model exhibits an abelian gauge symmetry $U(1)_X$ with gauge potential A_X as in $C_3 = A_X \wedge w_X + \dots$, switching on gauge fluxes G_4 entails a field-dependent Fayet-Iliopoulos D-term of the form [7, 31, 40, 51]

$$\xi_X(G_4) \simeq \frac{1}{2\mathcal{V}_B} \int_{\hat{Y}_4} w_X \wedge J \wedge G_4 \quad (3.15)$$

with \mathcal{V}_B the volume of the F-theory base B . The D-term for G_4^X and G_4^λ therefore reads

$$\xi_X(G_4^X) \simeq -\frac{2}{\mathcal{V}_B} \int_{B_3} J \wedge \mathcal{F} \wedge (3\mathcal{W} - 5\bar{\mathcal{K}}), \quad \xi_X(G_4^\lambda) \simeq \lambda \int_{B_3} J \wedge \mathcal{W} \wedge \bar{\mathcal{K}}. \quad (3.16)$$

Clearly for the non-restricted $SU(5)$ model no such $U(1)_X$ D-term arises as the abelian symmetry is higgsed. We will have more to say about D-terms in section 4.3.

Flux quantisation in $SU(5) \times U(1)$ models

The general quantisation condition for fluxes in M/F-theory on a Calabi-Yau four-fold \hat{Y}_4 is given in eq. (3.2). A possible shift from integrality of the flux would be due to curvature

contributions encoded in $\frac{1}{2}c_2(\hat{Y}_4)$. From our explicit computation of the topological invariants for \hat{Y}_4 it follows that $\frac{1}{2}c_2(\hat{Y}_4)$ takes the form

$$\frac{1}{2}c_2(\hat{Y}_4) = \frac{5}{2}\mathcal{W} \wedge w_X \quad \text{mod } \mathbb{Z}. \quad (3.17)$$

Since $5w_X$ is an integer form, a *sufficient* condition for the quantisation of $G_4^X(\mathcal{F}) = \mathcal{F} \wedge w_X$ is

$$\frac{1}{5}\mathcal{F} + \frac{1}{2}\mathcal{W} \in H^2(B, \mathbb{Z}). \quad (3.18)$$

However, this condition may well be too strong. To do better we need an explicit basis $\{\tilde{\omega}_\alpha\}$ of the integer cohomology $H^4(\hat{Y}_4, \mathbb{Z})$. Given such a basis one requires that

$$\int_{\hat{Y}_4} \left(G_4 + \frac{1}{2}c_2(\hat{Y}_4) \right) \wedge \tilde{\omega}_\alpha \in \mathbb{Z}. \quad (3.19)$$

At a general level, i.e. for an $SU(n)(\times U(1))$ Tate model over a generic base B , we do not have a basis of $H^4(\hat{Y}_4, \mathbb{Z})$. What we can do at this stage is deduce *necessary* conditions on the fluxes by integrating $G_4 + \frac{1}{2}c_2(\hat{Y}_4)$ against those integral four-forms which we *have* constructed. These are first of all the elements in $H_{\text{vert.}}^{2,2}(\hat{Y}_4, \mathbb{Z})$ given by the product of two two-forms and second the four-forms Poincaré dual to the matter surfaces S_R^k , which are examples of non-factorisable four-forms.

In particular we can test for a possible shift in the quantisation condition due to $\frac{1}{2}c_2(\hat{Y}_4)$. by integrating (3.17) against the above-mentioned set of four-forms. In appendix D we find that the only potential half-integer contribution can be expressed as the following integral defined entirely on the base,

$$\frac{1}{2} \int_B \mathcal{W}^2 \bar{\mathcal{K}}. \quad (3.20)$$

If we now specialise to models which are smoothly connected to a Type IIB orientifold by imposing the constraint (2.33), the integral (3.20) is manifestly integer and there is no curvature induced shift in the quantisation condition. In particular, this guarantees that all chiral indices in table 1 are integer, as they must be, because the matter surfaces S_R^k are among the integer four-forms appearing in the constraint (3.19) and $\frac{1}{2}c_2(\hat{Y}_4)$ has no effect.

Note that also for smooth Weierstrass models as well as for models with at most $SU(2)/Sp(1)$ singularities, the contribution $\frac{1}{2}c_2(\hat{Y}_4)$ is always integer, as shown in [44]. In spite of this agreement we stress that our analysis above gives rise to *necessary* conditions because it has not been settled that the set of integral four-forms we used is sufficient.

Fluxes in generic $SU(5)$ models

As was shown in [22] by generalizing the analysis of [30], it is possible to construct a similar set of fluxes for an F-theory-model with generic $SU(5)$ -singularity. The flux G_4^λ is unaffected by the transition from the restricted to the generic model. On the other hand, the G_4^X -flux ceases to exist as a factorisable flux in $H_{\text{vert.}}^{2,2}(\hat{Y}_4)$. This is because in the $U(1)$ -restricted model it depends on the resolution divisor S , which is not present for

generic $SU(5)$ models. Physically, the transition between both geometries corresponds to Higgsing of the extra $U(1)$ and thus affects the explicit form of the associated $U(1)$ -fluxes. Nonetheless, one can describe an analogous gauge flux $G_4^X(\mathcal{P})$ as the pullback of factorisable classes from the ambient five-fold \hat{X}_5 into which the four-fold \hat{Y}_4 is embedded via the Tate hypersurface constraint. The resulting flux \hat{Y}_4 is then not of factorisable form. In formulae, the fluxes we obtain in this way are

$$\tilde{G}_4^X(\mathcal{P}) = -X \wedge Y \wedge \mathcal{P} + (Z + \bar{\mathcal{K}} - \tfrac{1}{5}(2, 4, 6, 3)_i E_i) \wedge \mathcal{P}|_{\hat{Y}_4}, \quad (3.21)$$

$$\tilde{G}_4^\lambda = \lambda (E_2 \wedge E_4 + \tfrac{1}{5}(2, -1, 1, -2)_i E_i \wedge \bar{\mathcal{K}}). \quad (3.22)$$

In order to define $\tilde{G}_4^X(\mathcal{P})$ we assume that the Tate polynomial coefficient $a_{6,5}$ factorises into e.g. $a_{6,5} \simeq \rho \tau$; \mathcal{P} denotes the class of ρ . Indeed it was shown in [30] that, in such a case, the ambient space intersection $\{x = 0\} \cap \{y = 0\} \cap \{\rho = 0\}$ describes a four-cycle in the Calabi-Yau four-fold, denoted by σ_ρ . The resulting flux is of $(2, 2)$ -form only on the sublocus in complex structure moduli space for which $a_{6,5} \simeq \rho \tau$. This is because, unlike fluxes in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$, $\tilde{G}_4^X(\mathcal{P})$ generates a superpotential whose critical locus precisely corresponds to $a_{6,5} \simeq \rho \tau$.

The intersections of XY with \mathcal{P} on the $SU(n)$ -resolution manifold are mostly the same as those of the divisor class S with \mathcal{P} on the $SU(5) \times U(1)$ -resolution manifold [22]. In particular, for intersection with two base divisors one finds that

$$\int_{X_5^{su_n}} XY \mathcal{P} \mathcal{B}_a \mathcal{B}_b = \int_{\hat{Y}_4^{su_n \times u_1}} S \mathcal{P} \mathcal{B}_a \mathcal{B}_b, \quad (3.23)$$

where we leave the pullback map implicit on the right hand side. The only difference occurs for the two expressions S^2 versus $\sigma_\rho \cdot \sigma_\rho$,

$$\int_{\hat{Y}_4^{su_n \times u_1}} (S \mathcal{P})^2 = - \int_{B_3} \bar{\mathcal{K}} \mathcal{P}^2, \quad \text{but} \quad \int_{\hat{Y}_4^{su_n}} \sigma_\rho \cdot \sigma_\rho = - \int_{B_3} \bar{\mathcal{K}} \mathcal{P}^2 + \int_{C_{34}} \mathcal{P}, \quad (3.24)$$

where C_{34} lies in the class $(3\bar{\mathcal{K}} - 2\mathcal{W})(4\bar{\mathcal{K}} - 3\mathcal{W})$. Then the chirality of the **10**-state is the same as in the above case, while the chirality of the single **5**-state is the weighted sum of the chiralities of the two **5**-states in the above case (see [22]). The major difference occurs for the $D3$ -tadpole contribution, which obtains an additional term scaling linearly with \mathcal{P} ,

$$Q_{\text{gauge}, \mathcal{P}}(G_4^X) = - \int_{B_3} (\bar{\mathcal{K}} - \tfrac{3}{5}\mathcal{W}) \mathcal{P}^2 + \tfrac{1}{2} \int_{C_{34}} \mathcal{P}. \quad (3.25)$$

One notes that an expression defined in terms of the square of the four-form flux depends linearly on \mathcal{P} . A linear rescaling of the \mathcal{P} thus has different effects than a linear rescaling of $G_4^X(\mathcal{P})$. It is this property which, upon comparison to the IIB-picture in section 4.2 as well the process of recombination in section 5, fixes the overall scaling of both $G_4^X(\mathcal{P})$ and G_4^λ .

3.3 Comparison of G_4 -Fluxes and Spectral Cover Fluxes

Before moving on to the Type IIB picture let us take some breath and compare the factorisable G_4 fluxes of the previous two sections to the gauge fluxes obtained via spectral

covers [52, 53]. Motivated by duality with the heterotic string, the spectral cover or Higgs bundle construction encodes the neighbourhood of a 7-brane with non-abelian gauge group - here $SU(n)$ [8, 32, 35]. Among the possible fluxes constructed in this way is a so-called universal gauge flux defined everywhere in complex structure moduli space. Note that the spectral cover approach is exact in models with heterotic dual - see e.g. [54, 55] for early comparisons of the moduli spaces in such situations - but yields only a semi-local description for more general fibrations.

Specifically, non-restricted, i.e. generic $SU(5)$ -models admit a Higgs bundle description based on an $SU(5)_\perp$ spectral cover in which the visible gauge group emerges as the commutant of $SU(5)_\perp$ in an underlying E_8 . The associated universal spectral cover flux depends on a single parameter. Indeed the flux G_4^λ given in (3.5) is the precise analogue of this universal $SU(5)_\perp$ spectral cover flux. This has already been observed in [21]. In fact both the chiral spectrum and the 3-brane tadpole match. This is possible because for non-restricted $SU(n)$ models these quantities localise on the $SU(n)$ divisor, which is correctly captured by the semi-local spectral cover. As we have seen, the spectral cover fluxes completely exhaust the set of factorisable fluxes in such models.

F-theory models for $SU(n)$ with $n = 4, 3, 2$, on the other hand, correspond to commutant structure groups $SO(10)$, E_6 and E_7 . For these no spectral cover exists, and the corresponding fluxes, e.g. in heterotic compactifications, are constructed by different means such as del Pezzo fibrations or via the parabolic construction [52]. This seems to be the underlying reason why for $SU(n)$, $n < 5$ no simple factorisable fluxes of the form G_4^λ have been found.

The local version of the restricted $SU(n) \times U(1)$ models is given by a split spectral cover [11, 14, 33] (for values of n where spectral covers exist), generalising heterotic constructions with abelian gauge groups [56–59]. Again to be specific, the $SU(5) \times U(1)$ models are the global extension of a split spectral cover with structure group $S[U(4) \times U(1)]_\perp$. Indeed, an extra class of universal fluxes arises, which has been matched in [22] with $G_4^X(\mathcal{F})$. These fluxes are *not* localised entirely on the $SU(5)$ brane as is obvious already from the fact that they induce a chiral spectrum for the $SU(5)$ singlets $\mathbf{1}_5$. Correspondingly quantities like this chiral index and the 3-brane tadpole do receive contributions in the restricted Tate model which are not correctly captured by the split spectral cover.

Just as an example, consider the split spectral cover fluxes in the form presented in [14] with the independent chiralities of $SU(5)$ charged matter of the form

$$\chi_{\mathbf{10}_1} = \mu \int_{C_{\mathbf{10}_1}} 5\bar{\mathcal{K}} - 4\mathcal{W} \quad \chi_{\mathbf{5}_3} = \mu \int_{C_{\mathbf{5}_3}} \bar{\mathcal{K}}. \quad (3.26)$$

It is a simple enough task to find a linear combination of $G_4^X(\mathcal{F})$ and G_4^λ which reproduces these topological indices. The flux in question corresponds to a combination with integer coefficients

$$G_4^\mu := \mu \left(-G_4^X(\bar{\mathcal{K}}) + 4 G_4^{\lambda=1} \right) = \mu \left(4 E_2 E_4 + ((S - Z - \bar{\mathcal{K}}) + (2, 0, 2, -1)_i E_i) \bar{\mathcal{K}} \right). \quad (3.27)$$

This flux induces precisely the chiralities listed above and, in addition, it induces a chirality of $\int_{C_{\mathbf{1}_{-5}}} -\bar{\mathcal{K}}$ for the state $\mathbf{1}_{-5}$.

4 Gauge Fluxes in Type IIB Orientifolds and their Match with F-theory

We are now in a position to address our main objective, a quantitative comparison between G_4 fluxes and their Type IIB counterparts. To this end we first classify all consistent, generic gauge fluxes in Type IIB models with $U(n) \times U(1)$ brane configurations. Specialising again to $n = 5$ we find an intriguing match with the gauge fluxes described previously on the F-theory side. We also comment on the role of massive $U(1)$ s and their associated fluxes.

4.1 Generic Flux Configurations in Type IIB $U(n) \times U(1)$ -Models

In this section we describe the set of consistent gauge fluxes of the Type IIB orientifold models under consideration which do not break the non-abelian gauge symmetry. The latter constraint implies that only gauge fluxes associated with the diagonal $U(1)_a$ symmetry and with the second $U(1)_b$ due to the extra brane-image pair are of relevance. We furthermore focus on fluxes which arise as the pullback of two-forms from the ambient space X_3 onto the brane divisors as these are the flux components that induce nontrivial chirality on the branes.

Recall that the brane set-up of the $SU(n) \times U(1)_a \times U(1)_b$ model under consideration consists of a stack of n branes on a divisor $W = \{w = 0\}$ (along with an image stack on \tilde{W}) and a single brane on a second divisor V (along with its image on \tilde{V}). By D7-brane tadpole cancellation the latter is in the class

$$V = 4 D_{O7} - \left[kW + (k+1)\tilde{W} \right], \quad |n = 2k+1, \quad (4.1)$$

$$V = 4 D_{O7} - k \left[W + \tilde{W} \right], \quad |n = 2k. \quad (4.2)$$

Again it turns out useful to express the divisor classes in terms of the combinations $V_{\pm} = V \pm \tilde{V}$. Then $V_+ = 8 D_{O7} - n W_+$, while $V_- = W_-$ in the n -odd case and $V_- = 0$, when n is even.

We reiterate that the diagonal $U(1)_a$ is massive even in absence of gauge fluxes [60, 61]. This is because the divisors W and \tilde{W} lie in different homology classes, i.e. $W_- \neq 0$, as emphasized in particular in the recent discussion [31]. For $n = 2k$, the extra $U(1)$ is massless because $V_- = 0$, while for $n = 2k+1$ with $V_- \neq 0$ also the second $U(1)$ is massive by itself. However, it is easy to see that the mass matrix is of rank 1 so that there is a massless linear combination of $U(1)$ s given by

$$U(1)_X = \frac{1}{2} (U(1)_a - n U(1)_b), \quad (4.3)$$

where the overall normalization has been chosen for later convenience.

Next, let us consider a general flux on this brane set-up. Given a 7-brane wrapping a holomorphic divisor D , the relevant, gauge invariant quantity is not the curvature of the $U(1)$ -bundle per se, but its sum with the pullback of the B-field onto the brane. We choose the symbol F to denote precisely this gauge invariant combination. More precisely,

$$F|_D = \frac{\ell_s^2}{2\pi} \langle dA \rangle + B^{(+)}|_D \in H^2(D, \mathbb{Z}/2) \quad (4.4)$$

in terms of the orientifold even, discrete piece of the B -field B^+ with components 0 or $\frac{1}{2}$. Note that it is this combination that enters all topological quantities such as induced brane charges and chiral indices. We will comment on the relevance of the orientifold-odd component B^- in section 4.3.

We denote by F_a the so-defined flux on the $SU(n)$ divisor stack and by F_b the flux on the additional brane. The orientifold action σ maps the flux F_a on W to the image flux $F'_a = -\sigma^* F_a = -\tilde{F}_a$ on \tilde{W} , and similarly for F_b . Oftentimes it is again useful to introduce the notation

$$F_a^\pm = \frac{1}{2} (F_a \pm \sigma^* F_a). \quad (4.5)$$

In particular, F^+ contains the pullback of the discrete B-field $B_+ = b^\alpha \omega_\alpha$, where ω_α span a basis of $H_+^2(X_3, \mathbb{Z})$ and $b^\alpha = 0$ or $\frac{1}{2}$.

In order for the flux configuration to lead to a consistent string vacuum, the induced D5-tadpole must vanish. In particular this guarantees a non-anomalous spectrum. Recall that in orientifold models with non-trivial $H_-^2(X_3)$, spanned by a basis ω_a of orientifold-odd two-forms, the induced D5-tadpole is proportional to

$$\Gamma_a = \int_{X_3} \omega_a \wedge \sum_i n_i (D_{+,i} \wedge F_i^- + D_{-,i} \wedge F_i^+). \quad (4.6)$$

Here $D_{\pm,i}$ labels the combinations $D_i \pm \tilde{D}_i$ of brane divisors and image divisors, each carrying a stack of n_i 7-branes and corresponding fluxes. D5-brane tadpole cancellation requires that all $h_-^{1,1}(X_3)$ components of Γ_a vanish. Applied to the specific brane configuration under consideration this amounts to the constraint

$$0 = n W_+ (F_a^- - F_b^-) + W_- (n F_a^+ + F_b^+), \quad |n = 2k + 1, \quad (4.7)$$

$$0 = n W_+ (F_a^- - F_b^-) + n W_- F_a^+, \quad |n = 2k, \quad (4.8)$$

where we have used the fact that the pullback of any involution-odd class to D_{O7} is zero.

It is immediately clear that the following flux choices do not induce any D5-tadpole (where we express each flux as a tuple (F_a, F_b)):

	$n = 2k + 1$	$n = 2k$	
$F_X :=$	$(\frac{1}{2n}F, -\frac{1}{2}F)$	$(0, \frac{1}{n}F)$	(4.9)
$F_Y :=$	(B^-, B^-)	(B^-, B^-)	
$F_\lambda :=$	$(\frac{2\lambda}{n}D_{O7}, 0)$	$(\frac{2\lambda}{n}D_{O7}, 0)$	

Here F and B^- are general elements of $H_+^2(X_3)$ and $H_-^2(X_3)$, respectively, whose pullback to the divisors represents the corresponding fluxes as described above. It is understood that all quantities are chosen in a manner consistent with the Freed-Witten quantization condition. Our normalisation has been picked in order to facilitate match with F-theory. It turns out that F_Y is entirely trivial – it does not contribute to the D3-tadpole, to any chiral index and to the D-term of the massless $U(1)_X$. This is clear because it can be absorbed into the orientifold odd component of the B -field since the same amount of B^- is switched

on along all branes in the setup. That the orientifold-odd component of the B -field does not enter any of the above topological quantities in consistent setups satisfying D7- and D5-brane tadpole cancellation has been demonstrated in [62], and we will comment on its contribution to the D-terms in section 4.3. On the other hand, F_X is the flux associated with the massless combination $U(1)_X$ given by (4.3) for odd n and, respectively, with the massless $U(1)_b$ on the additional brane for even n .

We will now show that the fluxes F_X and F_λ completely exhaust the *generically* possible D5-tadpole free flux configurations. To see this let us make the following general ansatz for the fluxes

$$\begin{aligned} F_a &= k_a D_{O7} + w_a^+ W_+ + G_a^+ + w_a^- W_- + G_a^-, \\ F_b &= k_b D_{O7} + w_b^+ W_+ + G_b^+ + w_b^- W_- + G_b^-, \end{aligned} \quad (4.10)$$

where the G_i^+ do not involve \tilde{K} and W_+ and the G_i^- do not involve W_- . Substituting these into the above expression for the D5-tadpole results in three conditions which constrain G_i^\pm and relate the w_i^\pm .

Of course for special properties of the intersection matrix extra solutions may exist; these, however, are not among the *generically* possible flux configurations. Sticking to the generic case the most general form of flux can then be written as the sum of two positive flux configurations:

$$F_{gen} = (F_a, F_b) = F_X + F_\lambda, \quad (4.11)$$

where λ depends on the k_i and the w_i and the flux appearing in F_X is the involution even piece of F_b . Here we have used the fact that $W_+ - W_- = 2\tilde{W}$ along with the restriction (2.35) that $D_{O7} W = \tilde{W} W$. We furthermore discard a possible contribution F_Y as this is indistinguishable from a trivial flux configuration.

4.2 Specialisation to Type IIB $U(5)(\times U(1))$ Models and Match with F-theory

Having classified the *generic* D5-tadpole free flux configurations we can compare these with the generically possible G_4 -fluxes in the F-theory picture. Our strategy to establish such a correspondence is by comparing the flux-dependent topological quantities on both sides. For definiteness we now specialise to the case of $SU(5)$ models, but the analysis could equally well be carried out for other values of $SU(n)$.

Chiralities in $U(5) \times U(1)$ models

We first turn to the chiral index of charged fields. Type IIB orientifold models with $SU(5) \times U(1)_a \times U(1)_b$ symmetry are discussed extensively in [62]. The matter content is listed in the left column of table 2, where the subscripts denote the $U(1)_a$ and $U(1)_b$ charges. From the general chirality formula for the bifundamental matter (\bar{N}_a, N_b) between two stacks of branes along D_a and D_b (and similar expressions for the antisymmetric matter in $10_{(2,0)}$ and symmetric matter in $1_{(0,2)}$, see e.g. [62]),

$$I_{ab} = - \int_{X_3} D_a \wedge D_b \wedge (F_a - F_b), \quad (4.12)$$

one deduces the chiral indices for F_λ and F_X as summarised in table 2. In particular F_Y does not induce any chirality.

State	Chirality under F_λ	Chirality under F_X
$\mathbf{10}_{(2,0)}$	$\frac{\lambda}{5} \int_{X_3} D_{O7} W_+^2$	$\frac{1}{10} \int_{X_3} D_{O7} W_+ F$
$\mathbf{5}_{(1,-1)}$	$-\frac{\lambda}{10} \int_{X_3} D_{O7} W_+^2$	$\frac{1}{10} \int_{X_3} (9D_{O7} W_+ - 6W_+^2) F$
$\mathbf{5}_{(1,1)}$	$-\frac{\lambda}{10} \int_{X_3} D_{O7} W_+^2$	$\frac{1}{10} \int_{X_3} - (10D_{O7} W_+ - 6W_+^2) F$
$\mathbf{1}_{(0,2)}$	0	$\frac{1}{10} \int_{X_3} -5 (12D_{O7}^2 - 17D_{O7} W_+ + 6W_+^2) F$

Table 2. Chiral index with respect to generic fluxes in Type IIB orientifolds.

These expression are to be compared with the chiral index induced by the universal flux G_4^λ and the $U(1)_X$ flux $G_4^X(\mathcal{F})$ as collected in table 1. Indeed the F-theory $U(1)_X$ charges are reproduced by the Type IIB charges if we set

$$U(1)_X = \frac{1}{2} (U(1)_a - 5 U(1)_b), \quad (4.13)$$

which is precisely the massless combination of abelian factors, see eq. (4.3). With the help of the definitions (3.12) of the curves on the F-theory side and the F-theory/Type IIB dictionary (2.25) and (2.24),(2.26) one observes that the chiral indices in tables 2 and 1 match precisely if we furthermore identify

$$G_4^\lambda \leftrightarrow F_\lambda, \quad G_4^X(\mathcal{F}) \leftrightarrow F_X = \left(\frac{1}{10} F, -\frac{1}{2} F \right) \quad \text{with} \quad F = \pi^*(\mathcal{F}). \quad (4.14)$$

D3-tadpole in $U(5) \times U(1)$ models

This picture receives additional support from comparison of the induced D3-tadpole charges. For general fluxes, the Type IIB flux induced tadpole $Q_{\text{gauge, IIB}}$ appearing in (2.36) is

$$Q_{\text{gauge, IIB}} = -\frac{1}{4} \sum_i n_i \left(\int_{D_i} F_i^2 + \int_{\tilde{D}_i} (\sigma^* F_i)^2 \right). \quad (4.15)$$

Specialising to F_X and F_λ as above, one can read off the following D3-charges

$$Q_{\text{gauge, IIB}} = -\frac{1}{2} \int_{X_3} \left[(D_{O7} - \frac{3}{5} W_+) F^2 + \frac{1}{5} \lambda D_{O7} W_+ F + \frac{2}{5} \lambda^2 D_{O7}^2 W_+ \right]. \quad (4.16)$$

These agree with the analogous charges (3.14) on the F-theory side.

The D-term in $U(5) \times U(1)$ models

The identification (4.13) allows us to match also the flux induced D-terms for the massless $U(1)_X$ symmetry. In Type IIB, the $U(1)_X$ Fayet-Iliopoulos term is the corresponding linear combination of D-terms for the diagonal $U(1)_a$ and $U(1)_b$, i.e.

$$\xi_X \simeq \frac{1}{2\mathcal{V}_X} \left(\int_{X_3} D_a \wedge J \wedge F_a - 5 \int_{X_3} D_b \wedge J \wedge F_b \right). \quad (4.17)$$

Evaluated for the combinations F_X and F_λ this reproduces the F-theory D-terms (3.16).

Quantisation condition in $U(5) \times U(1)$ models

Finally we turn to flux quantisation in Type IIB orientifolds. According to the analysis of Freed and Witten for a single brane in oriented Type II string theory, the curvature $\langle dA \rangle$ of what is sloppily referred to as the $U(1)$ -bundle on the brane is half-integer quantised whenever the brane worldvolume is non-spin and integer otherwise [63]. This is usually generalised in the literature by demanding that the flux on each single brane satisfy this constraint individually. Applied to the case at hand, this reasoning leads to the two independent constraints

$$\left(\frac{1}{10}F + \frac{2}{5}\lambda D_{O7} + B + \frac{1}{2}W \right)|_W \in H^2(W, \mathbb{Z}), \quad (4.18)$$

$$\left(-\frac{1}{2}F + B + \frac{1}{2}W \right)|_W \in H^2(W, \mathbb{Z}), \quad (4.19)$$

where the first constraint relates to the $U(5)$ brane stack and the second is due to the fluxes along the $U(1)$ -brane $V = 4D_{O7} - 2\tilde{W} - 3W$. Recall that in our conventions the gauge flux is defined as the gauge invariant combination (4.4) including the B^+ -field, and in addition we must allow for a non-trivial B^- -field corresponding to the flux F_Y in (4.9). In effect, the full B -field appears in the quantisation condition.

While we do not present a complete evaluation of the resulting constraints, we comment on the possibility of a shift in flux quantisation due to the geometric contribution $\frac{1}{2}W$. In section 3.2 we argued that in F-models with smooth orientifold limit no such shift occurs, at least not from integration against the subset of four-forms of which we have explicit control. Therefore, if W is non-spin, i.e. if $\frac{1}{2}W$ is not an integer class, the Type IIB cousin of an F-theory $SU(n) \times U(1)$ restricted Tate model in the form considered in this article must involve a non-trivial discrete B -field such as to cancel the non-integer contribution. This discrete B -field then entirely translates into the geometry of the F-theory four-fold as probed by $c_2(\hat{Y}_4)$. This is also consistent with the findings of [44] for smooth models and $SU(2)/Sp(1)$ models. We reiterate, though, that the explicit quantisation results which have been derived in section 3.2 in F-theory are only necessary conditions because we have no complete basis of elements in $H^4(\hat{Y}_4)$ which are not in $H^{2,2}(\hat{Y}_4)$.⁶

Generic $U(5)$ models

Generic $U(5)$ -models are related to $U(5) \times U(1)$ models by a recombination process involving the $U(1)_b$ brane along divisor V and its image along \tilde{V} . The resulting invariant brane in the class V_+ is of Whitney type [41]. We will carefully analyse this recombination process in appendix B.2. A set of gauge fluxes along the Whitney brane are described in the language of $D9/\bar{D}9$ tachyon condensation, as discussed in [30, 41, 44] for models with at most $SU(2)/Sp(1)$ gauge groups. For such setups the Type IIB fluxes were identified with the analogue of the non-factorisable flux $G_4^X(\mathcal{P})$.

⁶Note, however, that if $\frac{1}{2}c_2(\hat{Y}_4)$ does receive non-integral contributions which we have not been able to detect with our method, then match with the two independent constraints (4.18) would not be possible. We therefore conjecture that this is not the case.

In appendix B.2 we generalise this to all $U(n)$ models. As an important novelty that becomes relevant for $U(n)$ models with $n > 2$, we show that the Whitney brane, despite being invariant, ‘knows’ about the V_- -contribution: While the $D7$ -charge only depends on the involution-invariant part of the $D7$ -charges of the brane/image-brane pair, the $D5$ - and $D3$ -charges also depend on the anti-invariant part. For example, the D3-brane tadpole-contribution of the above flux set-up in the $U(5)$ -case is

$$Q_{\text{gauge,IIB}} = -\frac{1}{2} \int_{X_3} P^2 (D_{O7} - \frac{3}{5} W_+) + \frac{P}{2} (4D_{O7} - 3W_+) (3D_{O7} - 2W_+). \quad (4.20)$$

This neatly matches the 3-brane charge of $G_4^X(\mathcal{P})$ in generic F-theory $SU(5)$ models, eq. (3.25), if we identify $P = -\pi^* \mathcal{P}$.

4.3 Massive $U(1)$ s and their Fluxes in Type IIB and F-theory

To conclude this analysis we would like to address the question of geometrically massive $U(1)$ s in the present context and compare our findings with the discussion in [31, 64].

In addition to the abelian gauge symmetry (4.13) with massless gauge potential the Type IIB orientifold enjoys also a massive $U(1)$ -symmetry given by the orthogonal combination

$$U(1)_{X'} = \frac{1}{2} (5 U(1)_a + U(1)_b). \quad (4.21)$$

This is because both $U(1)_a$ and $U(1)_b$ individually receive a mass by what has been dubbed in [31] the geometric Stückelberg mechanism. This Stückelberg mechanism operates even in absence of gauge flux due to the gauging of the shift symmetry of the axionic fields c^a obtained by expanding $C_2 = c^a \omega_a$ with ω_a a basis of $H_-^{1,1}(X_3)$. Massive $U(1)$ -symmetries are known to give rise to global perturbative symmetries respected by the perturbative Yukawa couplings and broken only non-perturbatively by D-brane instantons.⁷ The F-theory uplift of such a massive $U(1)$ was discussed in [31] in terms of a non-harmonic, more precisely non-closed, two-form $w_{X'}$. This allowed for a detailed match of the Stückelberg mechanism in the framework of gauged supergravity. Based on this field theoretic agreement it was furthermore suggested that the uplift of the Type IIB gauge fluxes associated with such massive $U(1)$ -symmetries likewise involves non-harmonic forms with the important caveat that such non-harmonic fluxes have to satisfy the F-theory uplift of the Type IIB D5-tadpole cancellation condition. This condition relates various components of the fluxes, in particular the non-harmonic ones, to each other. Note that while harmonic G_4 -fluxes automatically satisfy this constraint, it could not be decided at a general level whether the converse is also true, i.e. whether every gauge flux associated with a massive $U(1)$ admits an alternative description in terms of harmonic forms which is equivalent to the proposed description in terms of non-harmonic forms plus the D5-tadpole condition.

Let us compare this picture to the results of the present work: One of our main findings has been to identify the generic D5-tadpole-free $U(1)_a$ flux F_λ with the *harmonic* universal flux G_4^λ . In view of the above discussion this is not in contradiction with the

⁷Recent investigations of M5/D3-instantons in this context include [34, 64–69].

proposal of [31]; rather, it demonstrates that - at least for the fluxes discussed here - the F-theory D5-tadpole condition does remove the non-harmonic parts of the fluxes such that an effective description in terms of harmonic fluxes *is* possible. It would be important to further investigate if this phenomenon applies to all F-theory fluxes associated with a geometrically massive $U(1)$ in the Type IIB picture.

Another interesting question concerns the D-terms of the massive $U(1)$. In Type IIB orientifolds, where the mass of the geometrically massive $U(1)$ is g_s -suppressed with respect to the Kaluza-Klein scale, the associated D-terms are kept in the effective field theory analysis. The field-dependent Fayet-Iliopoulos term of such a massive $U(1)$ receives a contribution from the gauge flux and also from the dynamical B-field moduli b^a . These arise by expanding $B = b^a \omega_a$ and combine with the above mentioned c^a into $h_-^{1,1}(X_3)$ chiral multiplets $G^a = c^a - \tau b^a$ with τ the axio-dilaton. Note that these moduli are present whenever a $U(1)$ -symmetry becomes geometrically massive because the geometric Stückelberg mechanisms hinges on $h_-^{1,1} \neq 0$. In the case at hand, the Type IIB D-term for $U(1)_{X'}$ is therefore the linear combination of the terms⁸

$$\xi_{X'} \simeq \frac{1}{2\mathcal{V}_X} \left(5 \int_{X_3} D_a \wedge J \wedge (F_a + B^-) + \int_{X_3} D_b \wedge J \wedge (F_b + B^-) \right). \quad (4.22)$$

In the unhiggsed phase of vanishing vacuum expectation values of all charged open string fields the D-term supersymmetry conditions read

$$\xi_X = 0, \quad \xi_{X'} = 0. \quad (4.23)$$

On the other hand, the F-theory D-term supersymmetry condition involves only the vanishing of ξ_X given by (3.15). So are we missing the extra D-term $\xi_{X'} = 0$? Indeed, the gauged supergravity analysis of [31] suggests the existence of a non-closed two-form $w_{X'}$ describing the massive $U(1)_{X'}$ in F-theory. Dimensional reduction of the M-theory action including this non-harmonic form precisely reproduces the D-term $\xi_{X'}$. To understand why the constraint $\xi_{X'} = 0$ never appears in F-theory we must take into account that in Type IIB theory it is always possible to achieve $\xi_{X'} = 0$ by a suitable choice of B-field moduli b^a , see [64] for a more detailed discussion. Due to their appearance in the D-term these moduli acquire a mass of the order of the $U(1)_{X'}$ mass. In F-theory, where this mass is no longer suppressed with respect to the Kaluza-Klein scale, both the gauge boson and the B-field moduli have been integrated out. Therefore, the usual F-theory description which ignores the D-term $\xi_{X'}$ corresponds to the Type IIB configuration with the B-field moduli canceling the flux contribution to the D-term. If, on the other hand, we insist on including the B-field moduli into the effective action we also need to take into account the massive $U(1)_{X'}$ gauge boson. This leads to the dimensional reduction of [31] including the non-harmonic two-form $w_{X'}$.

⁸In ξ_X , eq. (4.17), the b^a -dependent terms have cancelled because of the specific homological relation between D_a and D_b which renders the combination $U(1)_X$ massless.

5 Brane Recombination

In this section we shed some more light on the brane recombination process that interpolates between the $U(1)$ -restricted F-theory model with $a_6 = 0$ on the one hand and the generic $SU(n)$ model on the other. We have seen in section 2 that the $U(1)$ -restricted and non-restricted models correspond to IIB brane set-ups with a brane/image-brane pair in the former and a Whitney-type brane in the latter case. The intermediate process of recombination is expected to be describable as a smooth deformation in the complex structure moduli space in F-theory, as described in this specific context in [30, 40].

Brane recombination is known to lead to a jump in the flux quanta [70]. In fact, since the overall $D3$ -charge is to be invariant under such a smooth deformation, the change in the flux should compensate precisely for the change in the geometrically induced $D3$ -charge. This has been exploited e.g. in [41] in similar contexts. From the analysis around eq. (2.10) we know that the change in the geometric $D3$ -brane charge is given by

$$\Delta Q_{D3}^g = \frac{1}{8} \chi(C_{34}). \quad (5.1)$$

The four-form fluxes, on the other hand, can be expressed as

$$G_4^{su_n} = \tilde{G}_4^X(\mathcal{P}) + \tilde{G}_4^\lambda, \quad G_4^{su_n \times u_1} = G_4^X(\mathcal{F}) + G_4^{\lambda'} \quad (5.2)$$

and their $D3$ -tadpole contribution is given in (3.14) and (3.25), respectively. Both expressions are integrals over the base manifold, which is the same for both resolution manifolds. It is therefore possible to simply subtract one of the expressions from the other. Let us define

$$A = \mathcal{P} - \mathcal{F}, \quad \sigma = \lambda - \lambda'. \quad (5.3)$$

Since the resulting expression for ΔQ_{gauge} should only depend on the differences of the fluxes, A and σ , not on the specific choice of \mathcal{P} , \mathcal{F} or λ , λ' , this fixes

$$A = \frac{1}{2} [a_{6,5}] = -\frac{1}{2} c_1(C_{34}), \quad \sigma = \frac{1}{2}. \quad (5.4)$$

Substituting these values back into the expression for ΔQ_{gauge} confirms that this contribution does indeed cancel ΔQ_{D3}^g .

The values for the change in the fluxes fits in nicely with the quantisation conditions. Since mod 2 we have $c_2^{su_n} = \frac{5}{2} \tilde{G}_4^\lambda$, while $c_2^{su_n \times u_1} = \frac{5}{2} G_4^X(W)$ (see the discussion in section 3.2), the following configurations are *sufficient* to meet the Freed-Witten quantisation condition:

$$\begin{aligned} SU(n) \times U(1) : \quad & \frac{1}{5} \mathcal{F} + \frac{1}{2} W \in \mathbb{Z}, & \frac{\lambda'}{5} \in \mathbb{Z}, \\ SU(n) : \quad & \frac{1}{5} \mathcal{P} \in \mathbb{Z}, & \frac{\lambda}{5} + \frac{1}{2} \in \mathbb{Z}. \end{aligned} \quad (5.5)$$

We observe that the change in the fluxes as given by (5.3) interpolates between these two configurations.

In field theory the recombination process is interpreted as a D-flat vacuum expectation value of vector-like open string recombination modes $\mathbf{1}_n + \mathbf{1}_{-n}$ localised on the recombination curve C_{34} . Before analyzing the condition for such vector-like pairs to exist, let us note that the flux $\tilde{G}_4^X(\mathcal{P})$ in the restricted and $G_4^X(\mathcal{F})$ in the unrestricted models differ in that $G_4^X(\mathcal{F})$ does not induce any superpotential in the effective action. This is because, being described by an element of $H_{\text{vert.}}^{2,2}(\hat{Y}_4)$, it is always of $(2,2)$ type. By contrast, $\tilde{G}_4^X(\mathcal{P})$ does induce a superpotential for the complex structure moduli of the four-fold. In Type IIB language, the superpotential fixes some of the brane deformation moduli describing the geometry of the Whitney brane.

This is consistent with the following picture: Before recombination there exist pairs of vector-like recombination modes $\Phi_i, \tilde{\Phi}_i$ localized on C_{34} which acquire a VEV in a D-flat manner if a_6 is switched on.⁹ The recombination modes participate in Yukawa or higher-order flux-dependent F-term couplings of the schematic form

$$\sum \Phi_i \tilde{\Phi}_j W_k(\zeta), \quad (5.6)$$

where ζ represent the open string moduli of the 7-brane. As the recombination moduli condense, the brane moduli ζ are constrained by a flux-dependent F-term. In particular if $W_k(\zeta)$ is quadratic, they acquire a mass after recombination.

The superpotential is such that after recombination $a_{6,5}$ factors into $a_{6,5} = \rho\tau$ [30]. As the two-form \mathcal{P} appearing in the flux $\tilde{G}_4^X(\mathcal{P})$ is in the class ρ it is constrained to lie in the domain

$$0 \leq \mathcal{P} \leq a_{6,5}. \quad (5.7)$$

This constraint admits a neat interpretation from the recombination picture: Together with the relations (5.3) and (5.4) between the fluxes before and after recombination, eq. (5.7) ensures that vector-like pairs of recombination modes transforming as $\mathbf{1}_n + \mathbf{1}_{-n}$ exist in the non-recombined phase $a_{6,5} = 0$. Put differently, fluxes outside the domain (5.7) would correspond to fluxes $G_4^X(\mathcal{F})$ in the non-recombined phase for which the spectrum of recombination modes on C_{34} is purely chiral. In this case no recombination in the above sense is possible and the flux $G_4^X(\mathcal{F})$ acts as an obstruction. (See [71] for a recent discussion of processes with chiral 'recombination' modes in the language of brane gluings).

This can be seen as follows: Since the massless modes in question are localized along the curve C_{34} the discussion can be phrased in terms of the line bundle L on C_{34} obtained by pulling back the gauge flux \mathcal{F} from the base B to C_{34} . In other words we define L as the line bundle on C_{34} with first Chern class or degree

$$d(L) = \mathcal{F}|_{C_{34}}. \quad (5.8)$$

The massless modes $\mathbf{1}_n$ and $\mathbf{1}_{-n}$ on C_{34} are given by the cohomology groups

$$H^i(C_{34}, L \otimes \sqrt{K_{C_{34}}}) \quad (5.9)$$

⁹In addition the spectrum generically contains a chiral excess of either Φ or $\tilde{\Phi}$ -fields.

with $i = 0$ and $i = 1$, respectively. The chiral index which follows by Riemann-Roch,

$$\chi(C_{34}, L \otimes \sqrt{K_{C_{34}}}) = h^0(C_{34}, L \otimes \sqrt{K_{C_{34}}}) - h^1(C, L \otimes \sqrt{K_{C_{34}}}) = d(L), \quad (5.10)$$

is in agreement with the result $\chi(\mathbf{1}_n) = \int_{C_{34}} \mathcal{F}$ as given in table 1 for the special case $n = 5$.

Now, a necessary condition for both $h^0(C_{34}, L \otimes \sqrt{K_{C_{34}}})$ and $h^1(C, L \otimes \sqrt{K_{C_{34}}})$ to be non-vanishing - corresponding to the existence of vector-like pairs of recombination modes - is that

$$-\frac{1}{2}d(K_{C_{34}}) \leq d(L) \leq \frac{1}{2}d(K_{C_{34}}), \quad (5.11)$$

where the degree $d(K_{C_{34}})$ is just

$$d(K_{C_{34}}) = -c_1(C_{34})|_{C_{34}} = a_{6,5}|_{C_{34}}. \quad (5.12)$$

First, by Kodaira's vanishing theorem, line bundles of negative degree on a curve cannot have any sections, hence for $h^0(C_{34}, L \otimes \sqrt{K_{C_{34}}})$ to be non-zero we need $0 \leq d(L \otimes \sqrt{K_{C_{34}}})$. By Serre duality, $h^1(L \otimes \sqrt{K_{C_{34}}}) = h^0(L^\vee \otimes \sqrt{K_{C_{34}}})$, from which the second inequality follows by the same argument.

Therefore, in order for brane recombination to be possible we need that

$$-\frac{1}{2}a_{6,5}|_{C_{34}} \leq \mathcal{F}|_{C_{34}} \leq \frac{1}{2}a_{6,5}|_{C_{34}}. \quad (5.13)$$

In view of our findings that $\mathcal{F} = \mathcal{P} - \frac{1}{2}a_{6,5}$, see (5.3) and (5.4), this is just the restriction of the constraint (5.7) onto C_{34} , as promised.

6 Summary and Open Questions

In this article we have established a detailed dictionary between G_4 gauge fluxes in F-theory compactifications and Type IIB orientifolds. We have focused on F-theory Tate models with $SU(n)$ and $SU(n) \times U(1)$ gauge groups for $n \leq 5$. The technology for resolving the associated Calabi-Yau four-folds generalises our methods in [22] and provides all necessary topological data of the F-theory geometry in a manner independent of the base space.

Key to our analysis of the Type IIB analogue of these F-theory compactifications has been to avoid the conifold singularity [32] typically encountered in the Sen limit of an F-theory model. In order for the $SU(n)$ F-theory model to be smoothly connected to a Type IIB orientifold certain topological restrictions on the base space B of the F-theory four-fold must be imposed. These restrictions amount to absence of an E_6 point in $SU(5)$ models and its analogue for other values of n . This demonstrates that, as expected, the class of F-theory compactifications is much larger than the class of Type IIB orientifolds.

Gauge fluxes in F-theory are given by G_4 -fluxes and are thus encoded in $H^{2,2}(\hat{Y}_4)$. The simplest such four-forms factorise into two two-forms and lie in the primary vertical subspace $H_{\text{vert}}^{2,2}(\hat{Y}_4)$. For the Tate models under consideration we have found two types of factorisable gauge fluxes in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$. First, in $SU(n) \times U(1)$ Tate models [40] the flux associated with the $U(1)$ -factor is of the form $G_4^X = \mathcal{F} \wedge w_X$ [22, 23, 30, 40]. Our analysis

identifies the corresponding Type IIB fluxes as a linear combination of flux on the $U(n)$ and the extra $U(1)$ -brane/image-brane stack. Second, for $SU(n)$ with $n = 5$ one finds one extra type of gauge flux in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$ called G_4^λ . This flux had been identified in [21] as the type of universal flux provided via spectral covers in the Higgs bundle approach. Our detailed match with Type IIB shows that this flux is associated with the diagonal $U(1) \subset U(5)$, which is massive by the geometric Stückelberg mechanism. This result is rather surprising given the very different group theory underlying both types of constructions. An important ingredient to establish the correspondence with Type IIB is the D5-tadpole cancellation condition, which is automatic for harmonic G_4 -fluxes but must be imposed by hand in Type IIB.

By deforming the $SU(n) \times U(1)$ model into a generic $SU(n)$ model, the factorisable flux G_4^X turns into flux given by an element of $H^4(\hat{Y}_4)$ which cannot be written as the sum of products of two-forms [22, 30]. It induces a superpotential that fixes some of the brane moduli. We have identified the related constraint on the flux quantum numbers as the condition for existence of vector-like pairs of recombination moduli. Furthermore, a careful analysis of the Whitney brane resulting from brane recombination in Type IIB establishes a quantitative match of the associated fluxes.

Apart from this type of non-factorisable fluxes, our analysis has focused on fluxes in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$. An important question for future work is to get a better handle on the remaining elements in $H^4(\hat{Y}_4)$ and the associated fluxes. As one of the most pressing open questions we point out that for $SU(n)$ models with $n < 5$, no universal fluxes of type G_4^λ exist as elements in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$. From the perspective of the spectral cover construction, this does not come as a complete shock. To understand this recall that for $SU(5)$, $G_4^\lambda \in H_{\text{vert}}^{2,2}(\hat{Y}_4)$ is the global extension of the spectral cover $SU(5)_\perp$ -flux, where $SU(5)_\perp$ describes the orthogonal complement of $SU(5)$ within E_8 . Gauge groups $SU(n)$, $n = 4, 3, 2$ would be associated with orthogonal structure groups $G = SO(10), E_6, E_7$, for which no spectral covers exist. Indeed in heterotic models, the corresponding flux is constructed by different methods such as del Pezzo fibrations or via extensions [52]. On the other hand, from a Type IIB perspective it is not obvious what distinguishes $SU(5)$ from $SU(n)$ with $n < 5$. Our result that the $SU(5)$ spectral cover fluxes represent merely the diagonal $U(1) \subset U(5)$ fluxes in Type IIB suggests that an analogue of the universal fluxes should exist also for $n < 5$. One possibility is that the missing fluxes in F-theory are not described by elements in $H_{\text{vert}}^{2,2}(\hat{Y}_4)$, but rather by non-factorisable fluxes. The corresponding four-forms must, however, exist as integral $(2, 2)$ forms for all values of complex structure moduli (unlike the ones obtained by recombination from G_4^X as sketched above) because the Type IIB fluxes do not induce a superpotential; this property should not change in F-theory.

A better understanding of non-factorisable four-forms is also required for a complete evaluation of the quantisation condition, which necessitates a basis of $H^4(\hat{Y}_4, \mathbb{Z}) \cap H^{2,2}(\hat{Y}_4)$. Once such a basis is available it will be possible to compare the F/M-theory quantisation condition in detail with the Freed-Witten condition in Type IIB. At least for F-theory models with a smooth Type IIB limit, our preliminary analysis suggests that the flux quantisation is not shifted by $\frac{1}{2}$ because $c_2(\hat{Y}_4)$ is even, in agreement with the findings of [44] for models with up to $Sp(1)/SU(2)$ singularities. If substantiated further, this result

would imply that the Type IIB limit of such models corresponds to configurations with non-trivial B -field such as to balance possible half-integer shifts in the Freed-Witten condition in case the brane divisors are non-spin. On the other hand, what is currently surprising is that the Type IIB Freed-Witten condition treats the flux on each brane independently, while G_4 fluxes in F-theory provide a combined description of all fluxes. The presence of extra four-forms could single out the flux integral over individual branes and render the quantisation prescriptions in both pictures equivalent. We hope to return to these questions in the future.

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A Detailed Geometry of F-theory Resolutions for $SU(n) (\times U(1))$ Models

In this appendix we provide the details of the resolution of singular Calabi-Yau four-folds \hat{Y}_4 describing $SU(n)$ and $SU(n) \times U(1)$ models in F-theory. Special emphasis will be put on the computation of intersection numbers involving the resolution divisors as well as topological quantities such as $c_2(\hat{Y}_4)$ the Euler characteristic $\chi(\hat{Y}_4)$.

A.1 Resolution Structure

$SU(n)$ -Models

Consider an elliptically fibered CalabiYau four-fold Y_4 given as a Weierstrass model in Tate form as in (2.1),

$$P_T = \{y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6\}. \quad (\text{A.1})$$

The Tate algorithm [9] identifies the vanishing orders of the sections a_i of $\bar{\mathcal{K}}^i$ along a divisor $\mathcal{W} : w = 0$ in B for a model with $SU(n)$ singularity along \mathcal{W} as follows:

$$\begin{array}{c|ccccc} & a_1 & a_2 & a_3 & a_4 & a_6 \\ \hline v_i^n & 0 & 1 & k & n-k & n \end{array} \quad (\text{A.2})$$

Here v_i stands for the vanishing order of $a_i = w^{v_i} a_{i,v_i}$, and k is defined via $n = 2k$ or $n = 2k + 1$ if n is even or odd respectively. Explicitly, we list the third and fourth vanishing orders for $n = 2, \dots, 5$ in table 3.

n	v_3^n	v_4^n
2	1	1
3	1	2
4	2	2
5	2	3

Table 3. Tate coefficients for $SU(n)$.

In [22] the resolution of such an $SU(n)$ singularity was worked out for $n = 5$. Here we generalise the procedure, which was inspired by the Tate algorithm [9], and explicitly apply it to $n = 2, 3, 4, 5$. The resolution manifold \hat{Y}_4 follows by a sequence of $n - 1$ blow-ups of the fibre over the singular divisor $\mathcal{W} : w = 0$. This blow-up induces $n - 1$ blow-up divisors d_k , where we let k run from 2 to n . Denoting by d_0 the proper transform of $w = 0$, the scaling relations induced by the blow-up can be shown to take the values given in table 4. See [22] for details of how to determine these scalings. One observes that the charges of x and y under the induced scaling relations are precisely given by the vanishing orders v_3^k and v_4^k respectively. This property will be connected in the forth-coming section on $SU(n) \times U(1)$ -models to the fact that for such models, the additional singularity resides on the curve $a_{3,v_3} = a_{4,v_4} = 0$.

k	x	y	d_0	d_k
2	1	1	1	-1
3	1	2	1	-1
4	2	2	1	-1
5	2	3	1	-1

Table 4. Scaling relations for $SU(n)$ resolution divisors.

The blow-up divisors each are \mathbb{P}^1 -fibrations over the divisor $w = 0$ in the base. Their intersection matrix reproduces the Cartan matrix of $SU(n)$. The order of the blow-up process ($n = 2, 3, 4, 5$), however, does not always co-incide with the intersection structure of these divisors. For example, in the $SU(5)$ -case the divisors intersect in the order $d_0, d_2, d_4, d_5, d_3, d_0$ - that is, d_0 intersects d_2 and d_3 but not d_4 or d_5 , etc.. One can then define coordinates e_i so as to represent the standard intersection structure

$$\int_{\hat{Y}_4} E_i E_j \mathcal{B}_a \mathcal{B}_b = C_{ij} \int \mathcal{W} \mathcal{B}_a \mathcal{B}_b, \quad (\text{A.3})$$

where C_{ij} takes the usual form of the $SU(n)$ -Cartan matrix and B_a are arbitrary divisors on the base. The identification between the d_n and the e_i for the various singularities is listed in table 5 (in all cases one defines $e_0 := d_0$).

	d_2	d_3	d_4	d_5
$SU(2)$	e_1			
$SU(3)$	e_1	e_2		
$SU(4)$	e_1	e_3	e_2	
$SU(5)$	e_1	e_4	e_2	e_3

Table 5. Labeling of resolution divisors.

Let us further denote by N the index of the divisor of the last blow-up. In the d_k -basis, this is simply $N = n = 2, 3, 4, 5$, while, in the e_i -basis, one finds from the above table that $N = 1, 2, 2, 3$ for the various singularities considered here. We will observe later on that the divisor e_N plays a special role in $SU(n) \times U(1)$ -models.

$SU(n) \times U(1)$ -Models

As described already in 2.1, for the $U(1)$ -restricted $SU(n) \times U(1)$ models with $a_6 \equiv 0$ an $SU(2)$ singularity appears in the fiber above the curve

$$C_{34} = \{a_{3,v_3} = 0\} \cap \{a_{4,v_4} = 0\}. \quad (\text{A.4})$$

This requires an additional resolution divisor $\{s = 0\}$ with scaling $(x, y, s) \sim (\lambda x, \lambda y, \lambda^{-1} s)$.

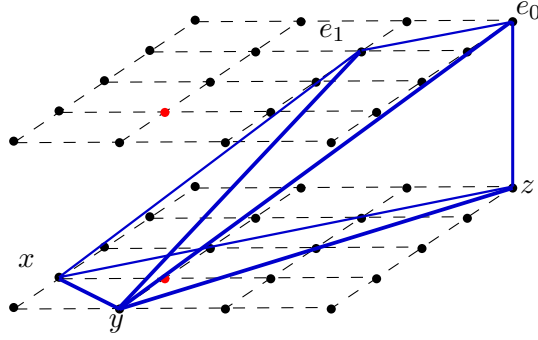


Figure 3. Toric Lattice Polytope corresponding to the sub-manifold of the $SU(2)$ -resolution manifold spanned by the coordinate sub-set $\{x, y, z, e_0, e_1\}$ (the lower red dot denotes the origin of the lattice)

A.2 Intersection Structure

$SU(n)$ -models

With these basic properties at hand, let us consider the intersection structure on the Calabi-Yau four-fold. The resolution blow-ups can be expressed in terms of toric geometry; in particular, the subspace spanned by the coordinates $\{x, y, z, e_0\} \cup \{e_i\}$ is a toric variety. This allows us to analyze part of the intersection structure by triangulating the polygon spanned by these variables and analyzing the resulting Stanley-Reisner ideal. This ideal encodes the set of variables which are not allowed to mutually vanish. Put differently, it encodes part of the patch structure of the ambient manifold, encoding e.g. that there is no patch on which x , y , and z may vanish. In order to deduce the intersection structure on the Calabi-Yau four-fold one has to furthermore take into account the proper transform of the Tate polynomial.

As an example let us consider the $SU(2)$ -case. There are two scaling relations,

$$(x, y, z, e_0, e_1) \sim (\lambda^2 x, \lambda^3 y, \lambda z, e_0, e_1) \sim (\sigma x, \sigma y, z, \sigma e_0, \sigma^{-1} e_1), \quad (\text{A.5})$$

and the polygon is subsequently given by

$$x = (-1, 0, \underline{0}), \quad y = (0, -1, \underline{0}), \quad z = (2, 3, \underline{0}), \quad e_0 = (2, 3, \underline{v}), \quad e_1 = (1, 2, \underline{v}). \quad (\text{A.6})$$

We can draw a 3-dimensional, schematic representation of the polygon, by mapping $\underline{0} \rightarrow 0$ and $\underline{v} \rightarrow 1$, see Figure 3. It is clear that all sides of the polygon except for one are triangular, and the remaining side allows for two triangulations. For each of the two triangulations we can then analyse the resultant cone structure, which encodes the patch structure of the corresponding toric variety. It is a simple matter to deduce that the following combinations of variables are not allowed to vanish simultaneously in the ambient space

$$\{xyz, xye_0, yze_1\} \otimes \begin{Bmatrix} xe_0 \\ ze_1 \end{Bmatrix}, \quad (\text{A.7})$$

where the choice between xe_0 and ze_1 is a choice between the two possible triangulations. Furthermore, the proper transform of the Tate polynomial becomes

$$y^2 + a_1xyz + a_{3,1}yz^3e_0 = x^3e_1 + a_{2,1}x^2z^2e_0e_1 + a_{4,1}xz^4e_0 + a_6z^6e_0^2. \quad (\text{A.8})$$

From this it is clear that, on the Calabi-Yau four-fold, both $x = e_0 = 0$ and $z = e_1 = 0$ imply $y = 0$. However, both xye_0 and zye_1 are not allowed to vanish simultaneously. Then on the Calabi-Yau four-fold both xe_0 and ze_1 become elements of the generator set of the SR-ideal, and, since these two generate the terms xye_0 and zye_1 , we can remove the latter two from the generator set. We thus arrive at the following set of SR-ideal generators, which is a subset of the total generator set: $\{xyz, ze_1, xe_0\}$.

	non-intersection divisors	N_{double}
$SU(2)$	$xyz, ze_{i i \geq 1}, xe_0$	1
$SU(3)$	$xyz, ze_{i i \geq 1}, xe_0, ye_0, ye_1$	3
$SU(4)$	$xyz, ze_{i i \geq 1}, xe_0, ye_0, xe_1, ye_1, xe_3, e_0e_2$	6
$SU(5)$	$xyz, ze_{i i \geq 1}, xe_0, ye_0, xe_1, ye_1, ye_2, xe_4, e_0e_2, T$ $T \in \{(e_0e_2, e_1e_3), (e_0e_2, e_2e_4), (e_1e_4, e_2e_4)\}$	9

Table 6. Part of the Stanley-Reisner ideal on the resolution four-folds.

In table 6 we list the generator set of combinations of variables whose zero loci do not intersect on the four-fold. Note that for $n = 5$ the polygon described above allows for 18 different triangulations, which reduce to three different Stanley-Reisner ideals on the four-fold. In the last column we count the number of elements which involve double intersections of two exceptional divisors, $E_i E_j$. For example, the divisor $\{x = 0\}$ lies in the divisor class $2(Z + \bar{\mathcal{K}}) - v_3^i E_i$, while the divisor $\{e_0 = 0\}$ lies in the class $\mathcal{W} - \sum_i E_i$. The vanishing of their intersection can thus be used to deduce

$$\sum_j v_3^j E_i E_j = 2\bar{\mathcal{K}} \sum_i E_i + \mathcal{W} v_3^i E_i - 2(Z + \bar{\mathcal{K}}) \mathcal{W}. \quad (\text{A.9})$$

We are interested in the number of linearly independant expressions for these double intersections in order to determine how many elements we can re-express in terms of base divisor intersections. By the above, xe_0 is one such relation, whilst, e.g. $ze_1 \rightarrow Z E_1 = 0$ does not involve any such double intersection.

In a next step, we would like to re-express two-forms of the form $E_i E_j$ as linear combinations of $E_i \bar{\mathcal{K}}$, $E_i \mathcal{W}$, $(Z + \bar{\mathcal{K}}) \mathcal{W}$ and possibly some remaining $E_i E_j$ -terms. The number of $E_i E_j$ -terms for the various $SU(n)$ -resolutions are $\binom{n}{2}$, i.e. 1, 3, 6, and 10 for $n = 2, 3, 4, 5$ respectively. From table 6 we thus see that for $n < 5$ these types of terms can be re-expressed entirely without any remaining $E_i E_j$ -terms, while one is left with one such term in the $SU(5)$ -case. Which combination of i and j one uses for this remaining term is a matter of choice. For later use, we choose a term that is orthogonal to the intersection of

(i, j)	C_{ij}	w_m	k_m
$(1, 1)$	-2	1	2

Table 7. Double-intersection coefficients for $SU(2)$ -model.

(i, j)	C_{ij}	w_m	k_m
$(1, 1)$	-2	$(2, 0)$	$(-1, 2)$
$(1, 2)$	1	$(-1, 0)$	$(2, -1)$
$(2, 2)$	-2	$(1, 1)$	$(-1, 2)$

Table 8. Double-intersection coefficients for $SU(3)$ -model.

(i, j)	C_{ij}	w_m	k_m
$(1, 1)$	-2	$(2, 0, 0)$	$(-1, 2, 2)$
$(1, 2)$	1	$(-1, 0, 0)$	$(1, -1, -1)$
$(1, 3)$	0	$(0, 0, 0)$	$(1, 0, 0)$
$(2, 2)$	-2	$(1, 1, 1)$	$(0, 2, 0)$
$(2, 3)$	1	$(0, 0, -1)$	$(-1, -1, 1)$
$(3, 3)$	-2	$(0, 0, 2)$	$(1, 2, 0)$

Table 9. Double-intersection coefficients for $SU(4)$ -model.

two base divisors because its corresponding entry in the Cartan matrix is zero. Specifically we choose $(i, j) = (2, 4)$.

The result of this analysis can be compactly written as

$$E_i E_j = C_{ij} (Z + \bar{K}) \mathcal{W} + w_m E_m \mathcal{W} + k_m E_m \bar{K} + b E_2 E_4, \quad (\text{A.10})$$

where $b = 0$ for $n < 5$. In tables 7 - 10 we list the coefficients of the expression for the various $SU(n)$ -cases. Note that in all cases the coefficients of the $(Z + \bar{K}) \mathcal{W}$ -term are the Cartan matrix entries C_{ij} . This becomes important in finding candidates for gauge fluxes.

$SU(n) \times U(1)$ -models

The resolution of the additional singularity along C_{34} , (A.4), also results in new generators of the Stanley-Reisner ideal. In addition to those elements listed in table 6 one finds

$$\{xy, zs, se_0\} \cup \{se_k \mid i \neq N\}. \quad (\text{A.11})$$

This implies that the relation (A.10) and in particular the coefficients listed in tables 7 - 10 carry over to the $U(1)$ -restricted case. Furthermore, one finds that the divisor $\{s = 0\}$ only intersects the Cartan divisor of the last blow-up,

$$S E_N = S \mathcal{W} \quad S E_i = 0 \quad \text{for } i \neq N. \quad (\text{A.12})$$

(i, j)	C_{ij}	w_m	k_m	b
(1, 1)	-2	(2, 0, 0, 0)	(-1, 2, 2, 2)	0
(1, 2)	1	(-1, 0, 0, 0)	(1, -1, -1, -1)	0
(1, 3)	0	(0, 0, 0, 0)	(0, 0, 0, 0)	0
(1, 4)	0	(0, 0, 0, 0)	(1, 0, 0, 0)	0
(2, 2)	-2	(2, 3, 0, 0)	(-2, -1, 2, 2)	-1
(2, 3)	1	(-1, -2, 0, 0)	(1, 2, -1, -1)	0
(2, 4)	0	(0, 0, 0, 0)	(0, 0, 0, 0)	1
(3, 3)	-2	(1, 2, 1, 1)	(0, -1, 2, 0)	1
(3, 4)	1	(0, 0, 0, -1)	(-1, -1, -1, 1)	-1
(4, 4)	-2	(0, 0, 0, 2)	(1, 2, 2, 0)	0

Table 10. Double-intersection coefficients for $SU(5)$ -model (for triangulation T_1).

Finally, the SR-ideal element xy implies the relation

$$6(Z + \bar{K})^2 - 5\bar{K}S + S^2 - (3v_3^i + 2v_4^i)E_i\bar{K} + nS\mathcal{W} + v_3^i v_4^j E_i E_j = 0. \quad (\text{A.13})$$

A.3 Topological Invariants

In order to analyse the D3-tadpole as well as the flux quantisation condition, we are interested in the second and fourth Chern class of the resolution manifold. It is a simple enough matter to deduce the former from that of the original manifold by adjunction. The relevant changes in the divisor classes during the resolution process can be read off from the induced scaling relations $(x, y, e_0, e_1) \sim (\lambda^{v_3}x, \lambda^{v_4}y, \lambda e_0, \lambda^{-1}e_1)$. It is given in terms of the powers of λ . Similarly the change in the class of the Tate polynomial simply depends on the sum of these terms. Then the Chern class of the resolution manifold at each blow-up step is given by:

$$\begin{aligned} \left[c \left(Y_4^{\text{aft}} \right) \right] &= \left[c \left(Y_4^{\text{bef}} \right) \right] \cdot (1 + E_i) \cdot \frac{1 + X^{\text{bef}} - v_3^i E_i}{1 + X^{\text{bef}}} \cdot \frac{1 + Y^{\text{bef}} - v_4^i E_i}{1 + Y^{\text{bef}}} \\ &\quad \cdot \frac{1 + E_0^{\text{bef}} - E_i}{1 + E_0^{\text{bef}}} \cdot \frac{1 + T^{\text{bef}}}{1 + T^{\text{bef}} - (v_3^i + v_4^i) E_i}, \end{aligned} \quad (\text{A.14})$$

where we have used T to denote the divisor class of the Tate divisor $\{P_T = 0\}$, and left the pullback map implicit for clarity of the expression. From this it is easy enough to derive the change in the second Chern class. We can then use the relations derived above to eliminate as many $E_i E_j$ -terms as possible from these expressions. The resulting expressions for the change in c_2 are summarised in table 11.

The situation for $SU(n) \times U(1)$ -models is similar. Since the resolution of the $SU(n)$ singularity is essentially unaffected by the restriction $a_6 = 0$, all we need to do is to evaluate

Gauge Group	Δc_2		
$Sp(1)/SU(2)$	$\mathcal{W} E_1$	$-7 \bar{\mathcal{K}} E_1$	
$SU(3)$	$(0, 2)_i \mathcal{W} E_i$	$-(4, 10)_i \bar{\mathcal{K}} E_i$	
$SU(4)$	$(1, 4, 1)_i \mathcal{W} E_i$	$-(6, 14, 8)_i \bar{\mathcal{K}} E_i$	
$SU(5)$	$(0, 2, 6, 2)_i \mathcal{W} E_i$	$-(4, 11, 17, 10)_i \bar{\mathcal{K}} E_i$	$+E_2 E_4$

Table 11. Change of second Chern class in $SU(n)$ -models compared to smooth Weierstrass models.

$SU(n)$	Δc_2			
$SU(2)$	U_2	$+2 \mathcal{W} E_1$	$-10 \bar{\mathcal{K}} E_1$	
$SU(3)$	U_3	$+(1, 4)_i \mathcal{W} E_i$	$-(6, 14)_i \bar{\mathcal{K}} E_i$	
$SU(4)$	U_4	$+(3, 8, 3)_i \mathcal{W} E_i$	$-(9, 20, 9)_i \bar{\mathcal{K}} E_i$	
$SU(5)$	U_5	$+(2, 6, 12, 3)_i \mathcal{W} E_i$	$-(6, 16, 24, 14)_i \bar{\mathcal{K}} E_i$	$+2 E_2 E_4$

Table 12. Change of second Chern class in $SU(n) \times U(1)$ -models compared to smooth Weierstrass models. Here $U_n = (6 \bar{\mathcal{K}} - n \mathcal{W}) (Z + \bar{\mathcal{K}} - S)$.

the change from the $SU(n)$ -resolution to the $SU(n) \times U(1)$ -resolution manifolds. This is given by

$$[c(Y_4^{su_n \times u_1})] = [c(Y_4^{su_n})] \cdot (1 + S) \cdot \frac{1 + X^{su_n} - S}{1 + X^{su_n}} \cdot \frac{1 + Y^{su_n} - S}{1 + Y^{su_n}} \cdot \frac{1 + T^{su_n}}{1 + T^{su_n} - S}, \quad (\text{A.15})$$

Adding this to the change from the original, non-singular manifold to the $SU(n)$ -manifold and eliminating as many $E_i E_j$ -terms as possible again, we find the change in the second Chern class from the non-singular manifold to the $SU(n) \times U(1)$ -resolution manifolds as summarized in table 12.

As is described in section 2.1, the change in the Euler characteristic can be computed from the change in the second Chern class by making use of the relation

$$\Delta c_4 = 3 (\Delta c_2)^2. \quad (\text{A.16})$$

This holds, provided Δc_2 is orthogonal to the second Chern class of the original, singular manifold. From tables 11 and 12 and general considerations of the intersections structure, it is clear that this is the case. In table 13, we list the resulting expressions for the change of the Euler number of $SU(n)$ -resolution manifolds compared to the original, non-resolved manifold.

The change in the Euler characteristic from the $SU(n)$ -resolution manifold to the $SU(n) \times U(1)$ -resolution manifold, on the other hand, becomes

$$\Delta_{su_n \times u_1}^{su_n \times u_1} \chi(Y_4) = -3 \int_{B_3} (3 \bar{\mathcal{K}} - v_3^n \mathcal{W})(4 \bar{\mathcal{K}} - v_4^n \mathcal{W})(6 \bar{\mathcal{K}} - n \mathcal{W}). \quad (\text{A.17})$$

At this point the importance of the coefficients v_3, v_4 in the induced scaling relations of the $SU(n)$ -resolution becomes clear, as they can now be used to reexpress the above as

$$\Delta_{su_n \times u_1}^{su_n \times u_1} \chi(Y_4) = 3 \chi(C_{34}). \quad (\text{A.18})$$

Gauge Group	$\Delta\chi$
$Sp(1) / SU(2)$	$-6 \int_{S_{GUT}} (49 \bar{\mathcal{K}}^2 - 14 \bar{\mathcal{K}} \mathcal{W} + \mathcal{W}^2)$
$SU(3)$	$-24 \int_{S_{GUT}} (19 \bar{\mathcal{K}}^2 - 8 \bar{\mathcal{K}} \mathcal{W} + \mathcal{W}^2)$
$SU(4)$	$-12 \int_{S_{GUT}} (50 \bar{\mathcal{K}}^2 - 28 \bar{\mathcal{K}} \mathcal{W} + 5 \mathcal{W}^2)$
$SU(5)$	$-15 \int_{S_{GUT}} (50 \bar{\mathcal{K}}^2 - 35 \bar{\mathcal{K}} \mathcal{W} + 8 \mathcal{W}^2)$

Table 13. Change of Euler characteristic due to resolution of $SU(n)$ singularities.

A.4 Matching D3-brane Tadpoles in F-theory and Type IIB

In this appendix, we list the expressions for the geometric D3-tadpole of Type IIB $U(n)$ -configurations and compare them to the related F-theory expressions.

$SU(n)$

From the analysis in section 2.3, one finds the D3-tadpole to be of the form

$$24 Q_{D3}^g = \frac{1}{2} \left(\sum_{i=0}^3 a_i D_{O7}^{3-i} W_+^i - \frac{3}{4} n W_+ (W_+^2 - W_-^2) + 12 c_2(B_3) D_{O7} \right) \quad (\text{A.19})$$

with

$$a_i = (360, -147n, 21n^2, -(n^3 - n))_i. \quad (\text{A.20})$$

This expression $24 Q_{D3}^g$ is to be compared with $\chi(\hat{Y}_4)$ of the resolved F-theory four-fold. The Euler characteristic of the resolved F-theory geometries for the various $SU(n)$ -models, as listed in table 13, can be written as

$$\chi(\hat{Y}_4) = \sum_{i=0}^3 b_i \bar{\mathcal{K}}^{3-i} \mathcal{W}^i + 12 c_2(B_3) \bar{\mathcal{K}}. \quad (\text{A.21})$$

The coefficients b_i for the various cases of $SU(n)$, $n = 2, 3, 4, 5$ are collected in table 14.

	b_1	b_2	b_3	b_4
$Sp(1)/SU(2)$	360	$-147n$	$21n^2$	$-(n^3 - n)$
$SU(3)$	360	$-152n$	$21n^2 + n$	$-(n^3 - n)$
$SU(4)$	360	$-150n$	$21n^2$	$-(n^3 - n)$
$SU(5)$	360	$-150n$	$21n^2$	$-(n^3 - n)$

Table 14. Coefficients b_i appearing (A.21).

$SU(n) \times U(1)$

The details of the Type IIB $SU(n) \times U(1)$ differ for even and odd values of n as described in section 2.2.

Even n

The geometric tadpole contribution for the Sen limit of $SU(n) \times U(1)$ with even n is

$$24 Q_{D3}^g = n \left(\chi(W) + \chi(\tilde{W}) \right) + 2\chi_0(4D_{O7} - \frac{n}{2}W_+) + 4\chi(D_{O7}) \quad (\text{A.22})$$

$$= a_i D_{O7}^{3-i} W_+^i - \frac{3}{4}nW_+ (W_+^2 - W_-^2) + 12c_2(B_3)D_{O7}, \quad (\text{A.23})$$

where

$$a_i = (144, -48n, 6n^2, -(\frac{1}{4}n^3 - n))_i.$$

This can be compared with the contribution for non-restricted $SU(n)$ models, for which we have already derived a corresponding formula on the F-theory side (see (A.18)). The difference to the tadpole contribution for non-restricted models, (A.19), is

$$\Delta Q_{D3}^g = (216, -99n, 15n^2, -\frac{3}{4}n^3)_i D_{O7}^{3-i} W_+^i, \quad (\text{A.24})$$

while the difference in the F-theory case is given by

$$\Delta Q_{D3}^g = 3\chi(C_{34}) = (216, -99n, 15n^2, -\frac{3}{4}n^3)_i \bar{\mathcal{K}}^{3-i} \mathcal{W}^i. \quad (\text{A.25})$$

This matches perfectly upon using the usual identifications $\pi^*(\bar{\mathcal{K}}) = D_{O7}$ and $\pi^*(\mathcal{W}) = W_+$. Since the IIB and the F-theory description of the $SU(n)$ models match, this also shows agreement for their $SU(n) \times U(1)$ cousins.

Odd n

For odd values of n the geometric tadpole is

$$24 Q_{D3}^g = n \left(\chi(W) + \chi(\tilde{W}) \right) + 2\chi_0(4D_{O7} - n/2 W_+) + 4\chi(D_{O7}) \quad (\text{A.26})$$

$$= a_i D_{O7}^{3-i} W_+^i + 12c_2(B_3)D_{O7} + 6D_{O7}W_-^2 \quad (\text{A.27})$$

with

$$a_i = (144, -48n, 6n^2, -\frac{1}{4}(n^3 - n))_i.$$

We first note that the last term vanishes because the restriction of an involution-odd class to the orientifold plane is zero. Then the remainder can be compared to the contribution for non-restricted models. The difference to the tadpole contribution for non-restricted models, (A.19),

$$\Delta Q_{D3}^g = (216, -99n, 15n^2, -\frac{3}{4}(n^3 - n))_i D_{O7}^{3-i} W_+^i - \frac{3}{4}nW_+ (W_+^2 - W_-^2), \quad (\text{A.28})$$

agrees with the difference in the F-theory case,

$$\Delta Q_{D3}^g = 3\chi(C_{34}) = (216, -99n + 9, 15n^2 - \frac{3}{2}n - \frac{9}{2}, -\frac{3}{4}(n^3 - n))_i \bar{\mathcal{K}}^{3-i} \mathcal{W}^i. \quad (\text{A.29})$$

upon the usual identifications.

B Gauge Fluxes in F-theory and Type IIB

B.1 Universal F-theory Fluxes

Following the analysis in section 3.1, we list a basis of flux candidates from double intersections of exceptional divisors in table 15. These can be re-expressed in terms of Cartan

Gauge Group	Basis of Lin. Comb. of $E_i E_j$ -terms $\perp B_a B_b$	count
$SU(2)$	none	0
$SU(3)$	$E_{11} - E_{22}, E_{11} + 2 E_{12}$	2
$SU(4)$	$E_{13}, E_{11} - E_{22}, E_{11} - E_{33}, E_{12} - E_{23}, E_{11} + 2 E_{12}$	5
$SU(5)$	$E_{13}, E_{14}, E_{24}, E_{11} - E_{22}, E_{11} - E_{33}, E_{11} - E_{44},$ $E_{12} - E_{23}, E_{12} - E_{34}, E_{11} + 2 E_{12}$	9

Table 15. Basis of linear combinations of double intersections of exceptional divisors which are orthogonal to the intersection of two base divisors.

fluxes for all cases except $n = 5$, in which case one is left with one term of the form E_{ij} . This can be used to define a linear combination of fluxes which leaves the $SU(n)$ -gauge group intact,

$$G_4^\lambda = E_2 E_4 + \left(\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{2}{5}\right)_i E_i \bar{K}. \quad (\text{B.1})$$

In the $U(1)$ -restricted models, the additional divisor introduces a new class of potential fluxes which may be used in combination with Cartan fluxes to form a class of fluxes leaving the $SU(n)$ -gauge group intact. These combinations are listed for $n = 2, \dots, 5$ in table 16.

gauge group	linear combination \perp to all Cartan fluxes
non-sing.	(1) $\cdot (S - Z - \bar{K})$
$SU(2)$	$(1; \frac{1}{2}) \cdot (S - Z - \bar{K}; E_1)$
$SU(3)$	$(1; \frac{1}{3}, \frac{2}{3}) \cdot (S - Z - \bar{K}; E_1, E_2)$
$SU(4)$	$(1; \frac{1}{2}, 1, \frac{1}{2}) \cdot (S - Z - \bar{K}; E_1, E_2, E_3)$
$SU(5)$	$(1; \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{3}{5}) \cdot (S - Z - \bar{K}; E_1, E_2, E_3, E_4)$

Table 16. Linear combination of vertical two-forms orthogonal to all Cartan fluxes.

B.2 Whitney-branes in Type IIB and their Gauge Fluxes

In this section we derive the general expression for the D3-tadpole of a Whitney-brane. The analysis closely follows [30, 44], but additionally takes into account involution-odd contributions. In this framework, a $D7$ -brane arises from tachyon condensation of a $D9$ /anti- $D9$ -pair. Viewing the Whitney brane as the result of a recombination process in which a brane combines with its image brane, it can then be derived from two $D9/\bar{D}9$ -pairs,

which form the brane and its image. It turns out that the $D7$ -charge of such a set-up only depends on the involution-invariant $D7$ -charge of the brane/image-brane pair, but the $D5$ - and $D3$ -charge also depend on the anti-invariant part. In other words, the Whitney brane resulting from a recombination process remembers the anti-invariant contributions of the brane/image-brane pair. We will demonstrate below that this logic is indeed necessary to arrive at the match between the F-theory- and Type IIB-fluxes as proposed in the main body of this paper.

Let us start with a single, ‘ordinary’ brane resulting from tachyon condensation of the following $D9/\bar{D}9$ -set-up:

$$\begin{array}{cc} D9_1 & \bar{D}9_2 \\ \mathcal{O}(D+E) & \mathcal{O}(E) \end{array} \quad (\text{B.2})$$

The charges of the resulting $D7$ -brane can be computed from

$$\Gamma_{D9} = (e^{D+E} - e^E) \cdot \left(1 + \frac{c_2}{24}\right) \quad (\text{B.3})$$

and, defining $F := E + \frac{1}{2}D$, are readily evaluated to be

$$(Q_{D9}, Q_{D7}, Q_{D5}, Q_{D3},) = (0, D, DF, \frac{1}{24}(D^3 + Dc_2) + \frac{1}{2}DF^2) \quad (\text{B.4})$$

Similarly, one can construct an image-brane by appropriately inverting the expression (B.2), resulting in the following charge vector:

$$\left(0, \tilde{D}, -\tilde{D}\tilde{F}, \frac{1}{24}(\tilde{D}^3 + \tilde{D}c_2) + \frac{1}{2}\tilde{D}\tilde{F}^2\right) \quad (\text{B.5})$$

where $\tilde{D} = \sigma^*D$ denotes the image of D under the orientifold involution.

Upon brane recombination, these two 7-branes will combine to form a singular brane, whose singularity resembles that of a Whitney umbrella. It is thus usually called a Whitney brane. Let us make the following notational conventions:

$$D_{\pm} = D \pm \sigma^*D, \quad [D_1 \cdots D_n]_{\pm} = (D_1 \cdots D_n) \pm \sigma^*(D_1 \cdots D_n) \quad (\text{B.6})$$

Then the charge vector of the Whitney brane, which is simply the sum of the charge vectors of the brane and its image, becomes

$$(0, D_+, [DF]_-, \frac{1}{24}([D^3]_+ + [D]_+c_2) + \frac{1}{2}[DF^2]_+). \quad (\text{B.7})$$

For a general $U(n)$ -configuration, the Whitney brane will have $D7$ -charge $8D_{O7-n}(W+\tilde{W})$. From the configuration we started with, (B.2), we know that D must be integer quantised. Then, if n is odd, it is not possible for D_- to be trivial unless W_- is trivial or W is spin.

In the case of a Whitney-brane without any fluxes, the only $D3$ -tadpole contribution should be of geometric type and depend on the Euler characteristic of the brane. Since the brane is singular, the usual formula for the Euler characteristic requires an adjustment, which was derived in [41] to be

$$\chi_0(D_+) = \int_{X_3} D_+^3 + c_2 \cdot D_+ + 3 D_+ D_{O7}(D_{O7} - D_+), \quad (\text{B.8})$$

Then we can split the $D3$ -charge into the geometric and the flux contributions as

$$Q_{D3}^g = \frac{1}{24}\chi_0(D_+) + Q_{D3}^{rem}, \quad (\text{B.9})$$

where the remainder term should vanish for a Whitney-brane without fluxes. This is achieved by the following two combinations

$$F^0 + \tilde{F}^0 = \pm (D_{O7} - \frac{1}{2}D_+), \quad F^0 - \tilde{F}^0 = \pm \frac{1}{2}D_-, \quad (\text{B.10})$$

where the overall sign must be the same in both expressions. Note that this automatically meets the $D5$ -tadpole condition.

In order to calculate the flux contribution to the $D3$ -charge of a Whitney brane, one should then use (B.9) and subtract the contribution of the F^0 -flux. This leads to the following expression for the flux contribution to the $D3$ -tadpole

$$Q_{D3}^f = \frac{1}{2}[DF^2]_+ - \frac{1}{8} \left\{ D_+ (D_+ - 2D_{O7}) - (D_-)^2 \right\} (F + \tilde{F}). \quad (\text{B.11})$$

A similar formula was derived in [30]; here we add the contribution of involution-odd fluxes. Note that there are again two flux configurations, for which this term becomes zero:

$$(F + \tilde{F}, F - \tilde{F}) = (0, 0), \quad (F + \tilde{F}, F - \tilde{F}) = (D_+ - 2D_{O7}, -D_-). \quad (\text{B.12})$$

In other words, the flux contribution to the $D3$ -tadpole vanishes precisely when the positive flux part lies in the trivial class or in that of $[a_{6,5}]$. This agrees with [30, 44], where it is noted that fluxes are only allowed on non-generic Whitney branes, in particular those for which $a_{6,5}$ factorises non-trivially.

Specialisation to $U(5)$ -Models

Let us apply the above analysis to the $SU(5)$ -setup described in section 4.1. Here the Whitney brane is such that $D_+ = 8D_{O7} - 5W_+$ and $D_- = W_-$, where W, \tilde{W} denote the divisor class of the GUT stack divisor and its image, respectively. We consider a flux configuration $\frac{1}{10}P$ on W , where $P \in H_+^2(X_3)$. $D5$ -tadpole cancellation then requires that the induced $D5$ -charge of the Whitney brane corresponds to setting $F = -\frac{P}{2}$ in the charge vector (B.7). Summing up the contribution of the $SU(5)$ and the Whitney brane to the $D3$ -brane tadpole eventually yields

$$-\frac{1}{2} \int_{X_3} P^2 (D_{O7} - \frac{3}{5}W_+) + \frac{P}{2} (4D_{O7} - 3W_+) (3D_{O7} - 2W_+). \quad (\text{B.13})$$

To arrive at this result we have used the relation $2D_{O7}W_+ = W_+^2 - W_-^2$.

C Intersections of Exceptional Divisors on $SU(5)$ -Resolution Manifold

In the following we list the triple and quadruple intersections of the exceptional divisors of the $SU(5)/SU(5) \times U(1)$ -resolution manifolds for the standard triangulation used in this and preceding paper, [22]. These can be derived from the Stanley-Reisner ideal elements in table 6 and are useful in the calculation of various flux-induced properties such as the $D3$ -tadpole for fluxes involving the wedge product of two dual forms of exceptional divisors.

We express the triple intersections as $E_{ijk} = (a, b)$, which we use as short-hand notation for

$$\int_{\tilde{Y}_4} E_i E_j E_k D = \int_{S_{GUT}} (a\bar{\mathcal{K}} + b\mathcal{W}) D. \quad (C.1)$$

The non-vanishing combinations are:

$$\begin{aligned} E_{111} &= (4, -4) & E_{112} &= (-3, 2) & E_{114} &= (-2, 0), \\ E_{122} &= (2, -1) & E_{124} &= (1, 0) & E_{222} &= (3, -4), \\ E_{223} &= (-4, 3) & E_{224} &= (-1, 0) & E_{233} &= (3, -2), \\ E_{234} &= (1, 0) & E_{244} &= (-1, 0) & E_{333} &= (-4, 1), \\ E_{334} &= (1, -1) & E_{344} &= (-2, 2) & E_{444} &= (2, -4). \end{aligned} \quad (C.2)$$

Similarly, we express the quadruple intersections as $E_{ijkl} = (a, b, c)$, which we use as short-hand notation for

$$\int_{\tilde{Y}_4} E_i E_j E_k E_l = \int_{S_{GUT}} a\bar{\mathcal{K}}^2 + b\bar{\mathcal{K}}\mathcal{W} + c\mathcal{W}^2. \quad (C.3)$$

The non-vanishing combinations are:

$$\begin{aligned} E_{1111} &= (-14, 20, 8), & E_{1112} &= (9, -12, -4), & E_{1114} &= (4, -4, 0), \\ E_{1122} &= (-6, 7, -2), & E_{1124} &= (-3, 2, 0), & E_{1144} &= (-2, 0, 0), \\ E_{1222} &= (4, -4, 2), & E_{1224} &= (2, -1, 0), & E_{1244} &= (1, 0, 0), \\ E_{2222} &= (-16, 29, -28), & E_{2223} &= (13, -22, 16), & E_{2224} &= (-1, 0, 0), \\ E_{2233} &= (-12, 17, -6), & E_{2234} &= (-1, 1, 0), & E_{2244} &= (0, -1, 0), \\ E_{2333} &= (9, -12, 4), & E_{2334} &= (3, -2, 0), & E_{2344} &= (-2, 2, 0), \\ E_{2444} &= (1, -2, 0), & E_{3333} &= (-8, 9, -4), & E_{3334} &= (-1, -1, 1), \\ E_{3344} &= (-2, 4, -2), & E_{3444} &= (4, -8, 4), & E_{4444} &= (-6, 12, -8). \end{aligned} \quad (C.4)$$

D Quantisation Condition

Here we consider the quantisation condition for $U(1)$ -restricted models, focussing for definiteness on $SU(5) \times U(1)$. Firstly, we note the following intersection properties for various powers of the resolution divisor S necessary to resolve the singularities over the curve C_{34} :

$$\int_{\hat{Y}_4} S(S + \bar{\mathcal{K}}) \mathcal{B}_a \mathcal{B}_b = 0, \quad \int_{\hat{Y}_4} S^2(S + \bar{\mathcal{K}}) \mathcal{B}_a = - \int_{C_{34}} \mathcal{B}_a, \quad \int_{\hat{Y}_4} S^2(S + \bar{\mathcal{K}})^2 = \chi(C_{34}). \quad (\text{D.1})$$

Then from $c_2(\hat{Y}_4) = 5w_X \wedge \mathcal{W} \bmod 2$ we find that the intersection $I_{AB} := \int_{\hat{Y}_4} A B c_2(\hat{Y}_4)$ with two divisor classes A and B take the following form mod 2:

$$I_{Z\mathcal{B}_a} = I_{ZE_i} = I_{ZS} = I_{\mathcal{B}_a\mathcal{B}_b} = I_{\mathcal{B}_aE_i} = I_{\mathcal{B}S} = I_{E_iS} = 0 \bmod 2, \\ I_{E_iE_j} = \begin{cases} 0 \bmod 2, \\ \int_{\mathcal{W}} \bar{\mathcal{K}} \mathcal{W} \bmod 2, \end{cases} \quad I_{SS} = \int_{\mathcal{W}} \bar{\mathcal{K}} \mathcal{W} \bmod 2. \quad (\text{D.2})$$

Here \mathcal{B}_a is a base divisor class, E_i are exceptional classes, and Z is the section $z = 0$. We observe that the only non-vanishing integral which one can obtain in this way takes the form $\int_{\mathcal{W}} \bar{\mathcal{K}} \mathcal{W}$. The constraint $2 \int_B \bar{\mathcal{K}}^2 \mathcal{W} = \int_B \bar{\mathcal{K}} \mathcal{W}^2$, which is necessary in order for an F-theory $SU(n)$ model to possess a smooth Type IIB limit, then implies that the integral of $c_2(\hat{Y}_4)$ with any of the above divisor classes is even. Similarly, no further restrictions arise from integrating $c_2(\hat{Y}_4)$ against the matter surfaces. This suggests that indeed no curvature induced shift in the quantisation condition occurs.

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